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FLEXURE WITH SHEAR AND ASSOCIATED TORSION IN PRISMS OF UNI-AXIAL AND ASYMMETRIC CROSS-SECTIONS

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CONTENTS

1. Introductory survey

It is now well over eighty years ago since Barré de Saint-Venant reduced the problem of the beam of constant cross-section under the action of a single transverse load to the search for plane harmonic functions satisfying a certain condition round the boundary of the cross-section. The solutions due to Saint-Venant, which include the rectangular, elliptic and circular cross-sections, are all cases in which the cross-sections have two axes of symmetry at right angles, meeting of necessity in the centroid of the cross-section, and along these axes the single transverse load is resolved. These axes are principal axes, and his solution depends upon this fact. Some less useful solutions exist for the load along one axis of certain beams of such bi-axial symmetry of cross-section, the solutions not yet being known for the load along the perpendicular axis.

In the case of cross-sections with a single axis of symmetry, when the forces across the end-section reduce to a load with a resolute transverse to the axis of symmetry, the solution of the flexure problem is known (Love 1927, p. 332) to involve also the solution of the Saint-Venant torsion problem of the cross-section. According to Young, Elderton and Pearson (1918, pp. 5, 58), Saint-Venant himself failed to realize this. The first exact solutions* for cross-sections of uni-axial symmetry involving this associated torsion were given by them in 1918 for complete and curtate circular sectors. These include a great variety of cross-sections, from some not far removed from the rectangular cross-section to the incomplete circular annulus or "gutter" section, the split cylindrical tube, and the circular cross-section with complete radial slit. The solution for the case with the load along the axis of symmetry, which involves no associated torsion, was later given by Seegar and Pearson (1920). There appears, unhappily, to be a complete neglect of the first of these works. Both Love (1927, p. 340) and Timoshenko (1934, p. 301), in referring to the case of flexure with associated torsion, unfortunately mention the work of Seegar and Pearson and do not give the proper reference, which should be to Young, Elderton and Pearson's paper. In spite of certain shortcomings of this paper to which the writer will in due course draw attention, the consequent apparent neglect of their work is ill-deserved, and the writer hopes that his own check upon and correction of some of their results will help to bring these earlier solutions for some very interesting cross-sections to more general notice. Their work was a war-time research bearing upon the torsion of aeroplane propellers, the Saint-Venant flexure problem being an admittedly crude approximation to the real problem of continuous loading involved.

One problem in which they went to considerable pains to proceed to a numerical conclusion was that for the bluff stream-line cross-section having as its boundary one loop of the lemniscate of Bernoulli, of polar equation $r^2 = 2c^2 \cos 2\theta$, for the case with the load transverse to the axis of symmetry. Here they found that their associated torsion was in the opposite sense to that for their solution for the right-angled circular sector having the same axis of symmetry and with its vertex at the origin $r = 0$. This transition of sign in their associated torsion, accompanying what appeared to be a not very marked change in the character of the cross-section, seemed somewhat doubtful to the writer, as indeed it did to the authors of the solution (Young, Elderton and Pearson 1918, p. 68), who published it "in the hope that a fresher eye may discover our slip, or verify the analysis, or again find other sections of negative torsion". Their solution for the lemniscate loop is now shown by the writer to be incorrect, the mistake of analysis located, a solution found by a different method and completed by finding the solution for the load along the axis of symmetry. Incidentally, the torsion problem for this cross-section is needed and a previous solution for the equivalent hydrodynamical problem, due to Basset (1884, p. 245), is also shown to be incorrect. Further, on tracing

* Griffith and Taylor (1921, p. 950; see also § 15) discuss approximate solutions for thin cross-sections, which include the split cylindrical tube.

the total twisting effect of the load upon the circular cross-sections, the writer disagrees with Young, Elderton and Pearson's interpretation of some of their results, in particular disliking their introduction of a so-called "total torsion", which is in fact not a pure torsion at all, and, further, has no meaning apart from uni-axial cross-sections.

In recent years further solutions for cross-sections of uni-axial symmetry have been published, one or two while this paper was in course of preparation. Shepherd (1932) has obtained the solution for a circular cross-section with a radial slit of any depth, and later (Shepherd 1936) the solution for the cardioid cross-section. In this latter paper he corrects an error of sign for the associated torsion in his earlier paper, but here also the writer finds some misinterpretation of otherwise correct results. It is worth mentioning that among Young, Elderton and Pearson's many results will be found the solution for the complete circular cross-section with a radial slit from centre to circumference, which links up with Shepherd's result.

Timoshenko (1922, p. 406) gives the solution when the cross-section is an isosceles triangle for particular values of Poisson's ratio, and also a simply obtained solution (Timoshenko 1934, p. 300) for a beam of semi-circular cross-section. This latter is incorrectly related to the external force system. He finds an incorrect position for the "flexural centre", i.e. the point on the axis of symmetry to which the load must be shifted from the centroid if the associated flexural torsion is omitted. This will be defined in § 2.

No account of recent work on Saint-Venant's flexure problem would be complete which did not pay tribute also to the work of Seth (1933) who first published a solution for the cross-section which is an isosceles right-angled triangle. This, in the writer's opinion, is incomplete in so far as he does not relate all the constants of his solution to the external force system; he does not in fact evaluate the associated flexural torsion.

The revision, completion, correction and correlation of the conflicting results made apparent by collecting these solutions together has been very much simplified by the writer's discovery that Saint-Venant's flexure problem is reducible to boundary problems of the same canonical simplicity as the torsion problem, involving six "canonical flexure functions" of which one is the torsion function for the cross-section.

Young, Elderton and Pearson remark (1918, p. 3) that "the problem of complete asymmetry still awaits investigation". With the aid of the six canonical functions the writer has carried out such a general investigation in the present paper, providing a systematic method of relating the necessary constants of a flexure solution to the external force system, expressing the associated flexural twist and the co-ordinates of the centre of flexure in terms of six "moment integrals" analogous to the torsion moment. Further, these have been worked out in a definite case for a particular asymmetric cross-section, namely, one of the two halves of the lemniscate loop previously mentioned, into which it is divided by its axis of symmetry.

2. Statement of the problem

The generators of a cylinder of constant cross-section S are parallel to the z-axis, and the curved surface of the cylinder is free from surface traction. The line of centroids of the cross-sections in the unstrained state is taken as the z-axis, and the x- and y-axes are for the present chosen to be the principal axes of the cross-section at the centroid G of the section $z = 0$. One end of the cylinder, $z = l$, is subject to forces which reduce to a single force transverse to the z-axis through the load-point $L(f', g')$ of this end-section. We shall suppose this force resolved as (W, W') parallel to these x- and y-axes (see fig. 1). The resultant of the stresses across the section $z = c$, $0 < c < l$, acting upon the material for which $z < c$, must then be equivalent to an equal force (W, W') at the load-point of this cross-section, together with a certain total bending moment about an axis transverse to the z-axis.

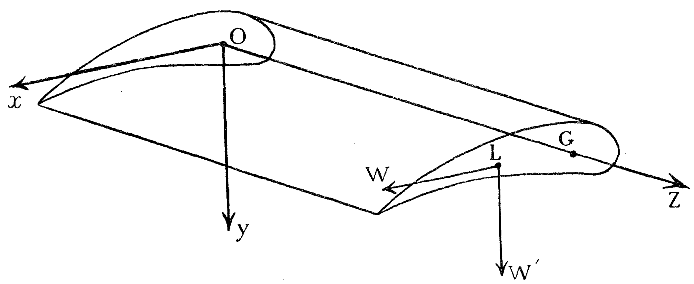

Fig. 1. Scheme of co-ordinate axes in relation to the elastic cylinder, and scheme of loading at the free end cross-section of the cylinder.

Note on use of elastic constants

We shall use the customary E for Young's modulus, and follow Karl Pearson in using η for Poisson's ratio and L. N. G. Filon in using σ for the modified Poisson's ratio of generalized plane stress, where $(1-\sigma)(1+\eta) = 1$, according as the results considered happen to be linear in η or σ respectively. (Although η is also used in certain sections of this work as a curvilinear co-ordinate, there will be found no reasonable likelihood of confusion arising from the two different uses of the same symbol.) The rigidity is written as μ.

Saint-Venant's solution of the flexure problem

The solution for the stresses in this problem can be written

$$\widehat{pq} = \widehat{pq}_1 + \widehat{pq}_2, \quad (p, q = x, y, z), \tag{2·1}$$

where
$$\widehat{xx} = \widehat{yy} = \widehat{xy} = 0, \tag{2·2}$$

and
$$\widehat{zz}_1 = (W/I)\, x(l-z), \tag{2·3}$$

$$\widehat{xz}_1 = (\mu W/EI)\left\{\frac{\partial \phi}{\partial x} - x^2 - \eta(x^2-y^2)\right\} + \mu\tau\left(\frac{\partial \phi_3}{\partial x} - y\right), \tag{2·4}$$

$$\widehat{yz}_1 = (\mu W/EI)\frac{\partial\phi}{\partial y}+\mu\tau\left(\frac{\partial\phi_3}{\partial y}+x\right), \tag{2·5}$$

$$\widehat{zz}_2 = (W'/I')\,y(l-z), \tag{2·6}$$

$$\widehat{xz}_2 = (\mu W'/EI')\frac{\partial\phi'}{\partial x}+\mu\tau'\left(\frac{\partial\phi_3}{\partial x}-y\right), \tag{2·7}$$

$$\widehat{yz}_2 = (\mu W'/EI')\left\{\frac{\partial\phi'}{\partial y}-y^2+\eta(x^2-y^2)\right\}+\mu\tau'\left(\frac{\partial\phi_3}{\partial y}+x\right), \tag{2·8}$$

in which I and I' are the principal second moments of the cross-section about the principal axes parallel to y and x respectively at 0; ϕ, ϕ', ϕ_3 are real plane harmonic functions satisfying the boundary conditions

$$\frac{\partial\phi}{\partial n} = l\{x^2+\eta(x^2-y^2)\}, \tag{2·9}$$

$$\frac{\partial\phi'}{\partial n} = m\{y^2-\eta(x^2-y^2)\}, \tag{2·10}$$

$$\frac{\partial\phi_3}{\partial n} = ly-mx, \tag{2·11}$$

in which n denotes the outward drawn normal to the boundary of direction cosines $(l, m, 0)$. In the case of ϕ_3 its conjugate function ψ_3 satisfies the boundary condition

$$\psi_3 = \tfrac{1}{2}(x^2+y^2)+\text{const.}, \tag{2·12}$$

and we find it convenient usually to find ψ_3 and deduce ϕ_3 if needed. The constants τ and τ' in (2·4), (2·5), (2·7), (2·8) are determined from the fact that, since the stresses across a section $z = c$, $(0<c<l)$, acting upon the material for which $z<c$, must be equivalent to the load (W, W') localized at (f', g'), in particular their moments about the z-axis must be the same, whence

$$\int(x\widehat{yz}_1-y\widehat{xz}_1)\,dS = -Wg', \tag{2·13}$$

$$\int(x\widehat{yz}_2-y\widehat{xz}_2)\,dS = W'f', \tag{2·14}$$

the integrals being taken over the cross-section S.

The flexure functions ϕ, ϕ' are not quite of the classical form of Saint-Venant's flexure functions χ, χ', as given in Love (1927, pp. 332, 343), but are closely related, since

$$\phi+i\phi' = -(\chi+i\chi')+(1+\tfrac{1}{2}\eta)\,(x+iy)^3/3. \tag{2·15}$$

The solution we are using is the form to which one is led naturally by the semi-inverse method, assuming (2·2), (2·3) and (2·6), and solving for the remaining stresses from the body-stress equations in the statical case under no body forces, and the consistency equations of type

$$(1+\eta)\,\nabla^2\widehat{pq}+\frac{\partial^2}{\partial p\,\partial q}\,(\widehat{xx}+\widehat{yy}+\widehat{zz})=0,\quad (p,q=x,y,z). \tag{2·16}$$

The stresses involving τ and τ' are torsional stresses, and equations (2·13) and (2·14) give the amounts τ and τ' of these torsional solutions associated with the load at the load-point $L(f',g')$. The rotation of an element of area in the plane of cross-section is

$$\omega=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right),$$

and from the stress-strain relations for the shears $\widehat{xz}$, $\widehat{yz}$ we have

$$\frac{\partial\omega}{\partial z}=\frac{1}{2\mu}\left(\frac{\partial\widehat{yz}}{\partial x}-\frac{\partial\widehat{xz}}{\partial y}\right)$$

or

$$\frac{\partial\omega}{\partial z}=\left(\tau-\frac{W\eta}{EI}y\right)+\left(\tau'+\frac{W'\eta}{EI'}x\right), \tag{2·17}$$

using (2·4), (2·5), (2·7) and (2·8). This will be taken as a measure of the local twist at a point (x, y) of a cross-section, and can be made to take any value by a suitable choice of the load-point. The local twist at the centroid of the cross-section is $\tau+\tau'$, and it is clear that this is also the mean value of the local twist taken across the cross-section.

Now by Saint-Venant's problem of flexure we usually understand the case in which the load-point L is taken at the centroid of the cross-section, i.e. $f'=g'=0$. The corresponding values τ_0, τ_0' given by (2·13), (2·14) will then be referred to as "*the* associated twists" simply. For a cross-section of uni-axial symmetry one of these is zero, and for a cross-section of bi-axial symmetry both vanish.

An alternative view of the problem of flexure is to take both τ and τ' zero in equations (2·4), (2·5), (2·7) and (2·8), and find the corresponding load-point (f_0', g_0') from (2·13) and (2·14). This point, i.e. the point of the cross-section which is the load-point when the local twist vanishes at the centroid of the section, or, what is the same thing, the load-point when the mean value of the local twists taken over the cross-section is zero, is called the "centre of flexure" or "flexural centre". It will lie on the axis of symmetry of a uni-axial cross-section and coincides with the centroid of a cross-section of bi-axial symmetry.

It is clear that the solution of the problem when the load-point is at the centroid corresponds to the superposition upon the solution for the load-point at the centre of flexure of a Saint-Venant torsion solution for the cross-section with a twist of amount $\tau_0+\tau_0'$. A solution of the flexure problem for a cross-section will not be regarded as complete unless either the associated twists τ_0, τ_0' or the co-ordinates of the centre of flexure (f_0', g_0') have been determined. Similarly a torsion problem cannot be regarded as complete until the torsion moment has been determined.

In the examples considered in the present paper we find both the associated twists and the position of the flexural centre, and give no detailed discussion of the stresses across the cross-sections; we concentrate upon relating completely the external force system applied to the free end of the cylinder with the constants of the solution, so that we have a more or less complete picture of the *general* action of a load applied at any point of the free end in twisting the cross-section at this end relative to that at the fixed end or "root" of the cylinder.

3. Subdivision of the flexure problem

The boundary conditions (2·9), (2·10) for ϕ and ϕ' can be combined in complex form as

$$\frac{\partial}{\partial n}(\phi+i\phi') = (1+\eta)(lx^2+imy^2)+i\eta(ily^2-mx^2). \tag{3·1}$$

Now
$$l=\frac{\partial x}{\partial n}=\frac{\partial y}{\partial s}, \quad m=\frac{\partial y}{\partial n}=-\frac{\partial x}{\partial s},$$

so that
$$lx^2+imy^2 = x^2\frac{\partial x}{\partial n}+iy^2\frac{\partial y}{\partial n}=\frac{\partial}{\partial n}(x^3+iy^3)/3$$

and
$$ily^2-mx^2 = x^2\frac{\partial x}{\partial s}+iy^2\frac{\partial y}{\partial s}=\frac{\partial}{\partial s}(x^3+iy^3)/3,$$

so that (3·1) can be written

$$\frac{\partial}{\partial n}(\phi+i\phi') = (1+\eta)\frac{\partial}{\partial n}(x^3+iy^3)/3+i\eta\frac{\partial}{\partial s}(x^3+iy^3)/3. \tag{3·2}$$

Hence if we write
$$\phi+i\phi' = (1+\eta)(\chi_0+i\chi_0')+i\eta(\phi_0+i\phi_0'), \tag{3·3}$$

where χ_0, χ_0', ϕ_0, ϕ_0' are real plane harmonic functions satisfying boundary conditions

$$\frac{\partial}{\partial n}\{\chi_0+i\chi_0'-(x^3+iy^3)/3\} = 0, \tag{3·4}$$

$$\frac{\partial}{\partial n}(\phi_0+i\phi_0')-\frac{\partial}{\partial s}(x^3+iy^3)/3 = 0, \tag{3·5}$$

then we have subdivided the flexure problem into separate problems, the boundary conditions for which are free of elastic constants and possess the same canonical simplicity of form as the boundary condition for the torsion function ϕ_3.

Further, if ψ_0, ψ_0' are the functions conjugate to ϕ_0, ϕ_0', so that

$$\frac{\partial}{\partial n}(\phi_0+i\phi_0') = \frac{\partial}{\partial s}(\psi_0+i\psi_0'),$$

then these satisfy the boundary conditions

$$\psi_0+i\psi_0'-(x^3+iy^3)/3 = \text{const.} \tag{3·6}$$

This replacement of the usual flexure functions χ and χ' by the four functions χ_0, χ_0', ϕ_0, ϕ_0' (or ψ_0, ψ_0'), has thus resulted in a considerably neater statement of the problem as a boundary problem than the classical one, besides separating the problem in a manner which allows the dependence of the results upon Poisson's ratio η to be obtained very readily, a desirable feature in an elastic problem of interest to the engineer.

4. Change of origin in torsion and flexure problems

The next step is to remove the restriction hitherto imposed upon the axes of being the principal axes at the centroid of the cross-section. This is mathematically necessary, since the principal axes at the centroid are, in general, only the most convenient axes to take from the point of view of analytical description of the boundary when each is an axis of symmetry of the cross-section.

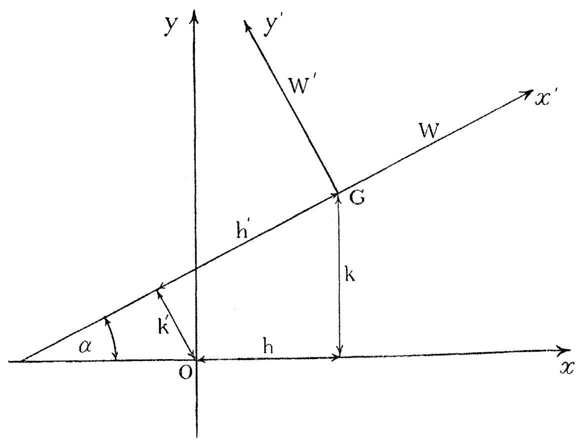

Fig. 2. Scheme for change of axes.

Let Gx', Gy' be the original axes of a cross-section (i.e. principal axes at the centroid G), and put $x'+iy' = \zeta$. Let Ox, Oy be any other rectangular axes (no longer necessarily principal axes), and let the angles which Gx', Gy' make with Ox, Oy respectively be α (see fig. 2). We put $x+iy = t$, with $t = t_0$ for the point $G\,(\zeta=0)$, and $\zeta = \zeta_0$ for the point $O\,(t=0)$, where

$$t_0 = h+ik, \quad -\zeta_0 = h'+ik'. \tag{4·1}$$

The equations of transformation from x, y to x', y' are given in complex form as

$$\zeta = (t-t_0)\,e^{-i\alpha}, \tag{4·2}$$

and we have

$$h'+ik' = (h+ik)\,e^{-i\alpha}. \tag{4·3}$$

Now if Ψ_3 denotes the torsion function in terms of co-ordinates referred to Gx', Gy', then the boundary condition (2·12) can be written

$$\Psi_3 = \tfrac{1}{2}\zeta\bar{\zeta}+\text{const.}, \tag{4·4}$$

where $\bar{\zeta}$ denotes $x'-iy'$, and generally throughout this paper a bar over a quantity is to denote the quantity obtained by changing i to $-i$.

In virtue of (4·2), (4·4) can be written

$$\Psi_3 = \tfrac{1}{2}t\bar{t}-\tfrac{1}{2}\bar{t}t_0-\tfrac{1}{2}t\bar{t}_0+\text{const.},$$

so that if we write
$$\Psi_3 = \psi_3 - hx - ky, \tag{4·5}$$

then ψ_3 is a real plane harmonic function and satisfies the boundary condition

$$\psi_3 = \tfrac{1}{2}t\bar{t}+\text{const.}, \tag{4·6}$$

which is of precisely the same *form* (4·4) satisfied by Ψ_3 in the original co-ordinates.

We derive easily
$$\Phi_3 = \phi_3 + hy - kx, \tag{4·7}$$

and, writing
$$\omega = \phi + i\psi,$$

$$\Phi_3 + i\Psi_3 = \omega_3 - i\bar{t}_0 t. \tag{4·8}$$

These give the torsion functions for the origin G in terms of the torsion functions for the origin O, and the form of the torsion problem is very little altered by change of axes as is well known.

Now consider the flexure problem in the same manner. We have, using (4·2),

$$\begin{aligned}(x'^3+iy'^3)/3 &= (\bar{\zeta}\zeta^2+\bar{\zeta}^3/3)/4\\ &= e^{-i\alpha}(\bar{t}-\bar{t}_0)\,(t-t_0)^2/4+e^{3i\alpha}(\bar{t}-\bar{t}_0)^3/12\\ &= \tfrac{1}{4}e^{-i\alpha}\{\bar{t}t^2-2t_0 t\bar{t}+t_0^2\bar{t}-\bar{t}_0(t-t_0)^2\}+e^{3i\alpha}(\bar{t}-\bar{t}_0)^3/12,\end{aligned} \tag{4·9}$$

and if we write

$$\chi_0 + i\chi_0 = e^{-i\alpha}[\chi_1+i\chi_2-t_0\chi_3+\{\bar{t}t_0^2-\bar{t}_0(t-t_0)^2-\bar{t}^3/3\}/4]+e^{3i\alpha}(\bar{t}-\bar{t}_0)^3/12, \tag{4·10}$$

where χ_1, χ_2, χ_3 are real plane harmonic functions, then

$$\begin{aligned}\chi_0+i\chi_0'-(x'^3+iy'^3)/3 &= e^{-i\alpha}\{\chi_1+i\chi_2-(\bar{t}t^2+\bar{t}^3/3)/4-t_0(\chi_3-\tfrac{1}{2}t\bar{t})\}\\ &= e^{-i\alpha}\{\chi_1+i\chi_2-(x^3+iy^3)/3-t_0[\chi_3-\tfrac{1}{2}(x^2+y^2)]\},\end{aligned} \tag{4·11}$$

so that the boundary condition (3·4) is satisfied if the functions χ_1, χ_2, χ_3 satisfy the boundary conditions

$$\frac{\partial}{\partial n}(\chi_1 - x^3/3) = 0, \tag{4·12}$$

$$\frac{\partial}{\partial n}(\chi_2 - y^3/3) = 0, \tag{4·13}$$

$$\frac{\partial}{\partial n}[\chi_3 - \tfrac{1}{2}(x^2+y^2)] = 0. \tag{4·14}$$

Similarly we may take real plane harmonic functions ψ_1, ψ_2, ψ_3 such that

$$\psi_0+i\psi_0' = e^{-i\alpha}[\psi_1+i\psi_2-t_0\psi_3+\{\bar{t}t_0^2-\bar{t}_0(t-t_0)^2-\bar{t}^3/3\}/4]+e^{3i\alpha}(\bar{t}-\bar{t}_0)^3/12, \quad (4\cdot15)$$

whence

$$\psi_0+i\psi_0'-(x'^3+iy'^3)/3 = e^{-i\alpha}\{\psi_1+i\psi_2-(x^3+iy^3)/3-t_0[\psi_3-\tfrac{1}{2}(x^2+y^2)]\}, \quad (4\cdot16)$$

and the boundary condition (3·6) is satisfied if ψ_1, ψ_2, ψ_3 satisfy the separate boundary conditions

$$\psi_1-x^3/3 = \text{const.}, \quad (4\cdot17)$$

$$\psi_2-y^3/3 = \text{const.}, \quad (4\cdot18)$$

$$\psi_3-\tfrac{1}{2}(x^2+y^2) = \text{const.} \quad (4\cdot19)$$

Also, if ψ_1, ψ_2, ψ_3 are the harmonic conjugates of functions ϕ_1, ϕ_2, ϕ_3, then, since

$$\frac{\partial\phi}{\partial t}=i\frac{\partial\psi}{\partial t}, \quad \frac{\partial\phi}{\partial\bar{t}}=-i\frac{\partial\psi}{\partial\bar{t}},$$

we find

$$\phi_0+i\phi_0' = e^{-i\alpha}[\phi_1+i\phi_2-t_0\phi_3-i\{\bar{t}t_0^2+\bar{t}_0(t-t_0)^2-\bar{t}^3/3\}/4]-ie^{3i\alpha}(\bar{t}-\bar{t}_0)^3/12. \quad (4\cdot20)$$

So the new flexure functions χ_1, χ_2, ψ_1, ψ_2 satisfy boundary conditions of precisely the same form as χ_0, χ_0', ψ_0, ψ_0', there being now no restriction upon the co-ordinate axes. These functions solve the problem when the axes are merely changed by rotation; if the origin is shifted from the centroid, then we need the solutions for the two further functions χ_3, ψ_3, of which the latter is clearly the torsion function for the cross-section. The boundary conditions (4·12), (4·13), (4·14), (4·17), (4·18), (4·19) are remarkably simple and symmetrical in form. We shall call the six functions the six "canonical flexure functions", and the six corresponding boundary conditions referred to above will be termed the "canonical boundary conditions".

Collecting up our solutions, equation (3·3) can now be written

$$\phi+i\phi' = (1+\eta)\{e^{-i\alpha}[\chi_1+i\chi_2-t_0\chi_3+\{\bar{t}t_0^2-\bar{t}_0(t-t_0)^2-\bar{t}^3/3\}/4]+e^{3i\alpha}(\bar{t}-\bar{t}_0)^3/12\}$$
$$+\eta\{e^{-i\alpha}[i(\phi_1+i\phi_2-t_0\phi_3)+\{\bar{t}t_0^2+\bar{t}_0(t-t_0)^2-\bar{t}^3/3\}/4]+e^{3i\alpha}(\bar{t}-\bar{t}_0)^3/12\}. \quad (4\cdot21)$$

5. The stresses

Next we consider the stresses in terms of the new co-ordinates and the canonical flexure functions. If $\widehat{xz}'$, $\widehat{yz}'$ are the shears across an element of the cross-section parallel to the principal axes Gx', Gy' at the centroid, and $\widehat{xz}$, $\widehat{yz}$ are the shears parallel to the new axes, we have

$$\widehat{xz}_1+i\widehat{yz}_1 = e^{i\alpha}(\widehat{xz}_1'+i\widehat{yz}_1'). \quad (5\cdot1)$$

Also we find

$$\frac{\partial}{\partial x'}+i\frac{\partial}{\partial y'} = 2\frac{\partial}{\partial\bar{\zeta}}, \quad \frac{\partial}{\partial x}+i\frac{\partial}{\partial y} = 2\frac{\partial}{\partial\bar{t}}. \quad (5\cdot2)$$

Hence from (2·4), (2·5) we have

$$(EI/\mu W)\,(\widehat{xz}_1'+i\widehat{yz}_1') = 2\frac{\partial\phi}{\partial\bar{\zeta}}-(1+\eta)\,(\zeta+\bar{\zeta})^2/4-\eta(\zeta-\bar{\zeta})^2/4+(EI\tau/W)\left(2\frac{\partial\Phi_3}{\partial\bar{\zeta}}+i\zeta\right), \tag{5·3}$$

and, using (4·2), this substituted into (5·1) gives

$$(EI/\mu W)\,(\widehat{xz}+i\widehat{yz}) = 2\frac{\partial\phi}{\partial\bar{t}}+(EI\tau/W)\left[2\frac{\partial\Phi_3}{\partial\bar{t}}+i(t-t_0)\right]$$
$$-(1+\eta)\,[(t-t_0)^2e^{-i\alpha}+2(t-t_0)\,(\bar{t}-\bar{t}_0)\,e^{i\alpha}+(\bar{t}-\bar{t}_0)^2\,e^{3i\alpha}]/4$$
$$-\eta[(t-t_0)^2e^{-i\alpha}-2(t-t_0)\,(\bar{t}-\bar{t}_0)\,e^{i\alpha}+(\bar{t}-\bar{t}_0)^2\,e^{3i\alpha}]/4. \tag{5·4}$$

But $$2\frac{\partial\phi}{\partial\bar{t}} = \frac{\partial}{\partial\bar{t}}\{(\phi+i\phi')+(\phi-i\phi')\};$$

hence from (4·21) we have

$$2\frac{\partial\phi}{\partial\bar{t}} = (1+\eta)\left\{2\frac{\partial}{\partial\bar{t}}[\cos\alpha\,\chi_1+\sin\alpha\,\chi_2-h'\chi_3]+[(t_0^2-\bar{t}^2)\,e^{-i\alpha}-2t_0(\bar{t}-\bar{t}_0)\,e^{i\alpha}+(\bar{t}-\bar{t}_0)^2\,e^{3i\alpha}]/4\right\}$$
$$+\eta\left\{2\frac{\partial}{\partial\bar{t}}[\sin\alpha\,\phi_1-\cos\alpha\,\phi_2+k'\phi_3]+[(t_0^2-\bar{t}^2)\,e^{-i\alpha}+2t_0(\bar{t}-\bar{t}_0)\,e^{i\alpha}+(\bar{t}-\bar{t}_0)^2\,e^{3i\alpha}]/4\right\}. \tag{5·5}$$

Also $$\Phi_3 = \phi_3-i\bar{t}_0t+it_0\bar{t}.$$

Hence (5·4) becomes

$$(EI/\mu W)\,(\widehat{xz}+i\widehat{yz}) = (EI\tau/W)\left(2\frac{\partial\phi_3}{\partial\bar{t}}+it\right)$$
$$+(1+\eta)\left\{2\frac{\partial}{\partial\bar{t}}[\cos\alpha\,\chi_1+\sin\alpha\,\chi_2-h'\chi_3]+[(2tt_0-t^2-\bar{t}^2)\,e^{-i\alpha}-2t(\bar{t}-\bar{t}_0)\,e^{i\alpha}]/4\right\}$$
$$+\eta\left\{2\frac{\partial}{\partial\bar{t}}[\sin\alpha\,\phi_1-\cos\alpha\,\phi_2+k'\phi_3]+[(2tt_0-t^2-\bar{t}^2)\,e^{-i\alpha}+2t(\bar{t}-\bar{t}_0)\,e^{i\alpha}]/4\right\}, \tag{5·6}$$

which separates to give

$$(EI/\mu W)\,\widehat{xz}_1 = (1+\eta)\left\{\cos\alpha\left(\frac{\partial\chi_1}{\partial x}-x^2\right)+\sin\alpha\frac{\partial\chi_2}{\partial x}-h'\left(\frac{\partial\chi_3}{\partial x}-x\right)\right\}$$
$$+\eta\left\{\sin\alpha\frac{\partial\phi_1}{\partial x}-\cos\alpha\left(\frac{\partial\phi_2}{\partial x}-y^2\right)+k'\left(\frac{\partial\phi_3}{\partial x}-y\right)\right\}+(EI\tau/W)\left(\frac{\partial\phi_3}{\partial x}-y\right), \tag{5·7}$$

$$(EI/\mu W)\,\widehat{yz}_1 = (1+\eta)\left\{\cos\alpha\frac{\partial\chi_1}{\partial y}+\sin\alpha\left(\frac{\partial\chi_2}{\partial y}-y^2\right)-h'\left(\frac{\partial\chi_3}{\partial y}-y\right)\right\}$$
$$+\eta\left\{\sin\alpha\left(\frac{\partial\phi_1}{\partial y}+x^2\right)-\cos\alpha\frac{\partial\phi_2}{\partial y}+k'\left(\frac{\partial\phi_3}{\partial y}+x\right)\right\}+(EI\tau/W)\left(\frac{\partial\phi_3}{\partial y}+x\right). \tag{5·8}$$

Proceeding similarly for $\widehat{xz}_2$, $\widehat{yz}_2$, or replacing I, W, τ, α by I', W', τ' and $\alpha+\pi/2$ respectively in (5·7), (5·8), we have

$$(EI'/\mu W')\,\widehat{xz}_2 = (1+\eta)\left\{-\sin\alpha\left(\frac{\partial\chi_1}{\partial x}-x^2\right)+\cos\alpha\frac{\partial\chi_2}{\partial x}-k'\left(\frac{\partial\chi_3}{\partial x}-x\right)\right\}$$
$$+\eta\left\{\cos\alpha\frac{\partial\phi_1}{\partial x}+\sin\alpha\left(\frac{\partial\phi_2}{\partial x}-y^2\right)-h'\left(\frac{\partial\phi_3}{\partial x}-y\right)\right\}$$
$$+(EI'\tau'/W')\left(\frac{\partial\phi_3}{\partial x}-y\right), \quad (5\cdot9)$$

$$(EI'/\mu W')\,\widehat{yz}_2 = (1+\eta)\left\{-\sin\alpha\frac{\partial\chi_1}{\partial y}+\cos\alpha\left(\frac{\partial\chi_2}{\partial y}-y^2\right)-k'\left(\frac{\partial\chi_3}{\partial y}-y\right)\right\}$$
$$+\eta\left\{\cos\alpha\left(\frac{\partial\phi_1}{\partial y}+x^2\right)+\sin\alpha\frac{\partial\phi_2}{\partial y}-h'\left(\frac{\partial\phi_3}{\partial y}+x\right)\right\}$$
$$+(EI'\tau'/W')\left(\frac{\partial\phi_3}{\partial y}+x\right). \quad (5\cdot10)$$

We have also $$(I/W)\,\widehat{zz}_1+i(I'/W')\,\widehat{zz}_2 = (l-z)\,(t-t_0)\,e^{-i\alpha}, \quad (5\cdot11)$$

all other stresses vanishing.

6. The displacements

We give the displacements here (apart from a possible rigid body displacement), mainly for the sake of completeness, although some use will be made of these results in §§ 11 and 12. They can be obtained by the usual method of integration via the stress-strain relations, or deduced from the known Saint-Venant solution (Love 1927, pp. 332, 343) in terms of the classical flexure functions χ, χ', since

$$\chi+i\chi' = -(\phi+i\phi')+(1+\tfrac{1}{2}\eta)\,\zeta^3/3. \quad (6\cdot1)$$

Writing the displacement vector $\mathbf{D}(u, v, w)$ in the form

$$\mathbf{D} = (W/EI)\,\mathbf{D}_1+(W'/EI')\,\mathbf{D}_2+(\tau+\tau')\,\mathbf{D}_3, \quad (6\cdot2)$$

we find $$(u_1+iv_1)+i(u_2+iv_2) = \eta(l-z)\,e^{-i\alpha}(t-t_0)^2, \quad (6\cdot3)$$

$$(u_1+iv_1)-i(u_2+iv_2) = e^{i\alpha}(lz^2-z^3/3), \quad (6\cdot4)$$

$$w_1+iw_2 = -(lz-\tfrac{1}{2}z^2)\,e^{-i\alpha}(t-t_0)$$
$$+(1+\eta)\,e^{-i\alpha}\{[\chi_1+i\chi_2-t_0\chi_3]-[\bar{t}(t-t_0)^2+(\bar{t}^3/3-\bar{t}t_0^2)]/4\}$$
$$+\eta e^{-i\alpha}\{i[\phi_1+i\phi_2-t_0\phi_3]+[\bar{t}(t-t_0)^2-(\bar{t}^3/3-\bar{t}t_0^2)]/4\}, \quad (6\cdot5)$$

$$u_3+iv_3 = iz(t-t_0), \quad (6\cdot6)$$

$$w_3 = \phi_3+hy-kx. \quad (6\cdot7)$$

Note that, if the form in terms of x and y is required in (6·5), we have

$$[\bar{t}(t-t_0)^2+(\bar{t}^3/3-\bar{t}t_0^2)]/4 = (x^3+iy^3)/3-\tfrac{1}{2}t_0(x^2+y^2), \tag{6·8}$$

$$[\bar{t}(t-t_0)^2-(\bar{t}^3/3-\bar{t}t_0^2)]/4 = (x^3+iy^3)/6+(xy^2+ix^2y)/2-\tfrac{1}{2}t_0(x^2+y^2)-\tfrac{1}{2}t_0^2(x-iy). \tag{6·9}$$

7. The associated flexural torsions

We now develop formulae for determining the amounts of the "associated flexural torsion" solutions, i.e. the associated twists τ, τ' defined in § 2, and given by equations (2·13), (2·14) with f', g' both zero. These we can obtain by proceeding directly, equating the sum of moments about Oz of the stresses across the cross-section to the moment of (W, W'), localized at the centroid, about Oz and separating the terms in W, τ (which vanish together) and W', τ', leading to

$$\int (x\widehat{yz}_1-y\widehat{xz}_1)\,dS = -Wk', \tag{7·1}$$

$$\int (x\widehat{yz}_2-y\widehat{xz}_2)\,dS = W'h'. \tag{7·2}$$

Multiply (7·1) by $(EI/\mu W)$ and substitute for $\widehat{xz}_1$ and $\widehat{yz}_1$ from (5·7) and (5·8), then we have

$$(1+\eta)\,[\cos\alpha\, L_1+\sin\alpha\, L_2-h'L_3] \\ +\eta[\sin\alpha\, M_1-\cos\alpha\, M_2+k'M_3]+(EI\tau/W) = -(EIk'/\mu), \tag{7·3}$$

where

$$\left.\begin{aligned} L_1 &= L_1'+\int x^2y\,dS, & M_1 &= M_1'+\int x^3\,dS,\\ L_2 &= L_2'-\int xy^2\,dS, & M_2 &= M_2'+\int y^3\,dS,\\ L_3 &= L_3', & M_3 &= M_3'+\int (x^2+y^2)\,dS,\end{aligned}\right\} \tag{7·4}$$

and

$$L_r' = \int\left(x\frac{\partial\chi_r}{\partial y}-y\frac{\partial\chi_r}{\partial x}\right)dS, \quad (r=1,\,2,\,3), \tag{7·5}$$

$$M_r' = \int\left(x\frac{\partial\phi_r}{\partial y}-y\frac{\partial\phi_r}{\partial x}\right)dS, \quad (r=1,\,2,\,3). \tag{7·6}$$

Substitute $E = 2\mu(1+\eta)$ in the right-hand side of (7·3), and the equation gives, using (4·3),

$$-\tau(EIM_3/W) = \cos\alpha[(1+\eta)\,(L_1-hL_3+2kI)-\eta(M_2-kM_3)] \\ +\sin\alpha[(1+\eta)\,(L_2-kL_3-2hI)+\eta(M_1-hM_3)], \tag{7·7}$$

which is the required formula for the associated twist τ.

In similar fashion (7·2) leads to

$$-\tau'(EI'M_3/W') = -\sin\alpha[(1+\eta)(L_1-hL_3+2kI')-\eta(M_2-kM_3)]$$
$$+\cos\alpha[(1+\eta)(L_2-kL_3-2hI')+\eta(M_1-hM_3)] \qquad (7\cdot 8)$$

for the associated twist τ'.

The twists are accordingly given in terms of six moment integrals defined by equations (7·4), (7·5), (7·6). Of these we are familiar with M_3, since $\mu\tau M_3$ is the torsion moment of the cross-section.

8. The torsion and flexure moment integrals

In this section we develop alternative expressions for the "moment integrals" L_r, M_r defined by equations (7·5), (7·6), which will be of considerable assistance in later sections.

Throughout the paper we shall use complex functions ω, Ω defined by

$$\omega_r = \phi_r + i\psi_r, \qquad (8\cdot 1)$$

$$\Omega_r = \chi_r + i\chi_r^*, \qquad (8\cdot 2)$$

where χ_r and χ_r^* are conjugate functions, and $r = 1$, 2 or 3. From this point onwards we shall use the more familiar notation $z = x+iy$ replacing the previous notation t for this complex variable, which we can very well do without causing any confusion, as there is very little, if any, reference to the z co-ordinates of points of the elastic cylinder hereafter.

Consider the integral $\int iz\frac{d\Omega_r}{dz}dS$, taken over the cross-section. It is readily shown that

$$iz\frac{d\Omega_r}{dz} = x\frac{\partial\chi_r}{\partial y} - y\frac{\partial\chi_r}{\partial x} + i\left(x\frac{\partial\chi_r}{\partial x} + y\frac{\partial\chi_r}{\partial y}\right),$$

so that L_r' is the real part of the integral, from (7·5), or

$$L_r' = R\int iz\frac{d\Omega_r}{dz}dS, \qquad (8\cdot 3)$$

and

$$M_r' = R\int iz\frac{d\omega_r}{dz}dS. \qquad (8\cdot 4)$$

Again since ϕ_r, ψ_r are conjugate functions

$$M_r' = -\int\left(x\frac{\partial\psi_r}{\partial x} + y\frac{\partial\psi_r}{\partial y}\right)dS,$$

which by Green's theorem, since also $\nabla^2\psi_r = 0$, becomes

$$M'_r = -\int \tfrac{1}{2}(x^2+y^2)\frac{\partial\psi_r}{\partial n}\,ds,$$

the line integral being taken round the boundary of the cross-section.

But
$$\frac{\partial\phi_r}{\partial s} = -\frac{\partial\psi_r}{\partial n};$$

hence finally we have
$$M'_r = \int \tfrac{1}{2}(x^2+y^2)\frac{\partial\phi_r}{\partial s}\,ds, \qquad (8\cdot5)$$

and similarly
$$L'_r = \int \tfrac{1}{2}(x^2+y^2)\frac{\partial\chi_r}{\partial s}\,ds. \qquad (8\cdot6)$$

Again these latter are equivalent to

$$M'_r = R\int \tfrac{1}{2}z\bar{z}\,d\omega_r, \qquad (8\cdot7)$$

$$L'_r = R\int \tfrac{1}{2}z\bar{z}\,d\Omega_r. \qquad (8\cdot8)$$

When $r = 3$, from (8·7) and the boundary condition (4·19) peculiar to ψ_3, we have

$$M'_3 = \int \psi_3\,d\phi_3. \qquad (8\cdot9)$$

Usually it will be found a great help to use one formula for a portion of the moment integral and another for the remainder.

9. The centre of flexure

The flexural centre is defined in § 2 as the load-point (f_0, g_0) when the local torsional twist at the centroid of the cross-section is zero, or, what amounts to the same thing, as the load-point when the mean torsion taken over the cross-section is zero. To find the co-ordinates of the centre of flexure we set τ, τ' zero in the stresses $\widehat{xz}$, $\widehat{yz}$ and then since the sum of the moments of the stresses across the cross-section about Oz is equivalent to the moment about Oz of the load (W, W') localized at (f_0, g_0), we have

$$\int (x\widehat{yz}_1 - y\widehat{xz}_1)\,dS = -W(g_0\cos\alpha - f_0\sin\alpha), \qquad (9\cdot1)$$

$$\int (x\widehat{yz}_2 - y\widehat{xz}_2)\,dS = W'(f_0\cos\alpha + g_0\sin\alpha). \qquad (9\cdot2)$$

Multiply (9·1) by $(EI/\mu W)$, substitute for $\widehat{xz}_1$ and $\widehat{yz}_1$ from (5·7) and (5·8) after putting $\tau = 0$, and substitute $E = 2\mu(1+\eta)$ in the right-hand side, then

$$(1+\eta)\,[\cos\alpha L_1+\sin\alpha L_2-h'L_3]+\eta[\sin\alpha M_1-\cos\alpha M_2+k'M_3] = -\,2I(1+\eta)\,(g_0\cos\alpha-f_0\sin\alpha)$$

or

$$2I(f_0\sin\alpha-g_0\cos\alpha) = \cos\alpha[(L_1-hL_3)-\sigma(M_2-kM_3)] +\sin\alpha[(L_2-kL_3)+\sigma(M_1-hM_3)], \tag{9·3}$$

where $\sigma = \eta/(1+\eta)$ is the modified Poisson's ratio of generalized plane stress, and proceeding similarly from (9·2) we have also

$$2I'(f_0\cos\alpha+g_0\sin\alpha) = -\sin\alpha[(L_1-hL_3)-\sigma(M_2-kM_3)] +\cos\alpha[(L_2-kL_3)+\sigma(M_1-hM_3)]. \tag{9·4}$$

Solving (9·3) and (9·4) for f_0, g_0, we have

$$f_0 = b_0[(L_2-kL_3)+\sigma(M_1-hM_3)]-h_0[(L_1-hL_3)-\sigma(M_2-kM_3)], \tag{9·5}$$

$$g_0 = h_0[(L_2-kL_3)+\sigma(M_1-hM_3)]-a_0[(L_1-hL_3)-\sigma(M_2-kM_3)], \tag{9·6}$$

where

$$a_0 = [(\cos^2\alpha)/I+(\sin^2\alpha)/I']/2, \tag{9·7}$$

$$b_0 = [(\sin^2\alpha)/I+(\cos^2\alpha)/I']/2, \tag{9·8}$$

$$h_0 = \sin\alpha\cos\alpha(1/I'-1/I)/2. \tag{9·9}$$

The introduction of σ as the elastic constant here is an advantage, since if the position of the flexural centre is known for the same cross-section for two different materials, by experiment for example, then its position for the same cross-section in any other material can be deduced by a simple linear interpolation. We note that since $0 \leqslant \eta \leqslant \frac{1}{2}$, we have $0 \leqslant \sigma \leqslant \frac{1}{3}$. Now equations (9·5) and (9·6) could be written in the form

$$f_0 = f_1+3\sigma(f_2-f_1), \tag{9·10}$$

$$g_0 = g_1+3\sigma(g_2-g_1), \tag{9·11}$$

where f_1, f_2, g_1, g_2 are constants for the cross-section and the chosen axes, and are independent of elastic constants; hence, whatever the elastic constant σ, the centre of flexure lies between the points (f_1, g_1), (f_2, g_2) and on the straight line joining them.

Before the formulae (9·5) and (9·6) can be employed, the principal moments I, I' at the centroid (h, k) and the angle α which the principal axes make with the co-ordinate axes must be determined. To this end let I_x, I_y, F be the second moments about the axes Ox, Oy respectively and the product moment about these axes. These will generally be the simplest moments to compute since the axes Ox, Oy will have been chosen for mathematical reasons of easy analytical description of the boundary of the cross-section. The corresponding quantities A, B, H for parallel axes at the centroid will then be given by

$$A = I_x-Sk^2, \quad B = I_y-Sh^2, \quad H = F-Shk, \tag{9·12}$$

where S is the area of the cross-section. We then find

$$\tan 2\alpha = 2H/(B-A), \tag{9·13}$$

$$I' = A\cos^2\alpha - 2H\sin\alpha\cos\alpha + B\sin^2\alpha,$$

$$I = A\sin^2\alpha + 2H\sin\alpha\cos\alpha + B\cos^2\alpha,$$

whence

$$2I' = A+B-C, \quad 2I = A+B+C, \tag{9·14}$$

$$2\cos^2\alpha = (B+C-A)/C, \quad 2\sin^2\alpha = (A-B+C)/C, \tag{9·15}$$

where

$$C^2 = 4H^2 + (B-A)^2. \tag{9·16}$$

Substituting these results in (9·7), (9·8) and (9·9), we have

$$a_0/A = b_0/B = h_0/H = 1/2(AB-H^2). \tag{9·17}$$

10. The centre of twist

The flexural centre gains an added importance from its practical relation to the "centre of twist" for the cross-section. If a twisting couple only is applied to the free end of the elastic cylinder, the point of the cross-section which suffers no displacement in the plane of cross-section is called the "centre of twist". The exact determination of this point depends on the conditions of fixity at the root. If the root is held completely rigid, however, it can be shown from the Rayleigh-Betti reciprocal theorem that the centre of twist is identical with the centre of flexure.

Duncan, Ellis and Scruton (1933)* have demonstrated, both theoretically and by experiment, that, if the cylinder is long and the root supports are "quasi-rigid", the two points cannot be very far from one another. (Some remarks bearing on their experiments will be found in § 12.) These ideas have found some use in the discussion of the sustained vibrations in aeroplane wings which characterize the dangerous phenomenon of "wing-flutter".

11. The simplifications of uni-axial symmetry

When the cross-section has an axis of symmetry this is a principal axis at all points along it, and in particular at the centroid of the cross-section, which must lie upon the axis of symmetry. The centroid is only rarely the most convenient origin of co-ordinates to adopt, as in such exceptional cases as the circular cross-section with a radial crack, or the equilateral triangular cross-section, soluble in a simple manner in the case when $\eta = \frac{1}{2}$.

With the x-axis as the axis of symmetry, we have $k = 0$, $\alpha = 0$, $k' = 0$, $h' = h$, and

* See also Southwell (1936, p. 29). Southwell omits the third author in citing this paper.

the flexural displacements are given in terms of the co-ordinates x, y, z by the equations $u = u_1 + u_2$ etc., where

$$(EI/W)\,u_1 = \tfrac{1}{2}\eta(l-z)\,\{(x-h)^2-y^2\}+lz^2/2-z^3/6, \tag{11·1}$$

$$(EI/W)\,v_1 = \eta(l-z)\,y(x-h), \tag{11·2}$$

$$(EI/W)\,w_1 = (1+\eta)\,\{\chi_1 - x^3/3 - h[\chi_3-\tfrac{1}{2}(x^2+y^2)]\} \\ -\eta\,\{\phi_2 - x^3/6 - xy^2/2 + h(x^2+y^2)/2 - h^2x/2\} - (lz-\tfrac{1}{2}z^2)\,(x-h), \tag{11·3}$$

$$(EI'/W')\,u_2 = \eta(l-z)\,y(x-h) - (\tau' EI'/W')\,yz, \tag{11·4}$$

$$(EI'/W')\,v_2 = -\tfrac{1}{2}\eta(l-z)\,\{(x-h)^2-y^2\}+lz^2/2-z^3/6+(\tau' EI'/W')\,(x-h)\,z, \tag{11·5}$$

$$(EI'/W')\,w_2 = (1+\eta)\,(\chi_2 - y^3/6) + \eta\,\{\phi_1 - h\phi_3 + y^3/6 + x^2y/2 - h^2y/2\} \\ +(\tau' EI'/W')\,(\phi_3+hy) - (lz-\tfrac{1}{2}z^2)\,y. \tag{11·6}$$

The stresses are given by

$$(EI/\mu W)\,\widehat{xz}_1 = (1+\eta)\left\{\frac{\partial\chi_1}{\partial x} - x^2 - h\left(\frac{\partial\chi_3}{\partial x} - x\right)\right\} - \eta\left\{\frac{\partial\phi_2}{\partial x} - y^2\right\}, \tag{11·7}$$

$$(EI/\mu W)\,\widehat{yz}_1 = (1+\eta)\left\{\frac{\partial\chi_1}{\partial y} - h\left(\frac{\partial\chi_3}{\partial y} - y\right)\right\} - \eta\frac{\partial\phi_2}{\partial y}, \tag{11·8}$$

$$(I/W)\,\widehat{zz}_1 = -(l-z)\,x, \tag{11·9}$$

and

$$(EI'/\mu W')\,\widehat{xz}_2 = (1+\eta)\frac{\partial\chi_2}{\partial x} + \eta\left\{\frac{\partial\phi_1}{\partial x} - h\left(\frac{\partial\phi_3}{\partial x} - y\right)\right\} + (EI'\tau'/W')\left(\frac{\partial\phi_3}{\partial x} - y\right), \tag{11·10}$$

$$(EI'/\mu W')\,\widehat{yz}_2 = (1+\eta)\left\{\frac{\partial\chi_2}{\partial y} - y^2\right\} + \eta\left\{\frac{\partial\phi_1}{\partial y} + x^2 - h\left(\frac{\partial\phi_3}{\partial y} + x\right)\right\} + (EI'\tau'/W')\left(\frac{\partial\phi_3}{\partial x} + x\right), \tag{11·11}$$

$$(I'/W')\,\widehat{zz}_2 = -(l-z)\,y. \tag{11·12}$$

The associated twist τ' occurring in these equations is given by

$$-\tau'(EI'M_3/W') = (1+\eta)\,(L_2 - 2hI') + \eta(M_1 - hM_3), \tag{11·13}$$

and the position of the centre of flexure by $(f_0, 0)$, where

$$f_0 = [L_2 + \sigma(M_1 - hM_3)]/2I'. \tag{11·14}$$

It will be seen that in the case of uni-axial symmetry of cross-section the canonical flexure functions fall into two groups, χ_1, χ_3, ϕ_2 corresponding to the resolute W of the load, and χ_2, ϕ_1, ϕ_3 corresponding to the resolute W'. But for the complete flexure problem it is still necessary to solve for the six canonical functions, so that intrinsically the general asymmetric cross-section should present no greater difficulties than are met with in the case of uni-axial cross-sections. The chief simplification arises in the calculation of the associated twist τ' and the co-ordinates of the centre of flexure, where we have only three moment integrals to compute, one of which is a multiple of the torsion moment for the cross-section.

12. Remarks on the measurement of twist by experiment

Duncan, Ellis and Scruton's determination (1933, p. 213) of the position of the centre of flexure rests on the fact that "if a constant flexural load in a direction at right angles to the axis of symmetry be applied at a number of points in the section and the twist is measured for each point of application, then the position of the flexural centre can be found, since the twist is proportional to the distance of the line of action of the load from that centre". The twist is apparently taken as proportional to the difference between deflexions of the leading and trailing edges of the uni-axial cross-sections examined. Let us examine what is really measured by such a procedure. The deflexion v for a load W' at the centre of flexure and any twisting couple $\mu\tau' M_3$, which are together equivalent to the load W' at some other point on the axis of symmetry, is given by

$$v = (W'/EI')\left[-\tfrac{1}{2}\eta(l-z)\{(x-h)^2-y^2\}+lz^2/2-z^3/6\right]+\tau'(x-h)\,z+\gamma x-\alpha z+\beta', \tag{12·1}$$

where the terms in γ, α and β' correspond to a small rigid body displacement necessitated by conditions of fixation at the root.

Now suppose that v is measured as $v(x_1)$, $v(x_2)$ at two points $(x_1, 0)$, $(x_2, 0)$ on the axis of symmetry $(x_2>x_1)$, as in Duncan, Ellis and Scruton's experiments for example, where the points are on the leading and trailing edges. The difference of deflexions is $v(x_2)-v(x_1)$, and the relative difference of deflexions between two cross-sections $z=z_1, z_2$ is $\Big[v(x_2)-v(x_1)\Big]_{z_1}^{z_2}$. If we divide this by x_2-x_1 we get an experimental "angle of twist" of the section z_2 relative to the section z_1, and dividing this again by z_2-z_1 we should have the corresponding angle of twist per unit length.

But (12·1) gives

$$\frac{\Big[v(x_2)-v(x_1)\Big]_{z_1}^{z_2}}{(x_2-x_1)\,(z_2-z_1)} = \tau'+(W'\eta/2EI')\,(x_1+x_2-2h), \tag{12·2}$$

which is not the theoretical angle of twist τ' per unit length, unless $h=(x_1+x_2)/2$, i.e. unless the points at which the deflexions are measured in the cross-sections are symmetrically disposed about the centroid of the cross-section.

It would appear that Duncan, Ellis and Scruton have not considered and allowed for this difference, and there seems no valid reason for neglecting the second term in (12·2), which arises from the "anticlastic displacements" (cp. Love 1927, p. 340) as we shall describe the terms in the displacements u, v independent of τ'. The experimental angle of twist per unit length should be determined from the relative deflexions of points on the axis of symmetry symmetrically disposed about the centroid of the cross-section. If this is not feasible in practice it must be deduced from (12·2).

13. Remarks on the so-called "total torsion"

In their work on uni-axial cross-sections, Young, Elderton and Pearson (1918, p. 19) term the quantity on the right-hand side of (12·2) the "total torsion". If their equations (20) for the displacements are compared with our equations (11·4), (11·5), (11·6) (remembering that they take the y-axis as the axis of symmetry), it will be found that their associated twist τ is equivalent to our $\tau' - W'\eta h/EI'$, and they call the term $(x_1+x_2)\,W'\eta/2EI'$ the "anticlastic torsion". This difference of outlook arises in the following way:

If we define anticlastic displacements for the origin $X = 0$, $Y = 0$ by the mathematical forms

$$(EI'/W')\,u = \eta(l-z)\,XY, \tag{13·1}$$

$$(EI'/W')\,v = -\tfrac{1}{2}\eta(l-z)\,(X^2-Y^2)+lz^2/2-z^3/6, \tag{13·2}$$

then anticlastic displacements relative to one origin are equivalent to anticlastic displacements relative to a second origin together with torsional and rigid body displacements. Young, Elderton and Pearson take their origin on one edge of the uni-axial section, the present writer takes the origin *for which the above forms of u and v hold* at the centroid of the cross-section. Their τ is the local twist at their origin, our τ' is the local twist at the centroid of the cross-section.

Their definition of the associated flexural twist relating it to an origin on one edge of the uni-axial cross-section, chosen largely for mathematical purposes, is not a very happy one. It does not lend itself to extension to the asymmetrical case, unless the cross-section has a definite edge, and even then from all points of view it seems much more satisfactory to retain the definition of the associated twist which relates it to the centroid, as we have done, since this point has the same physical significance for all cross-sections. Similarly in the case of the anticlastic displacements, these should be defined by the forms (13·1), (13·2) *for principal axes at the centroid*, or their equivalent for other axes by transformation of co-ordinates, and the writer deprecates the use of the word torsion at all in connexion with these anticlastic flexural displacements.

The "total torsion" is the apparent torsion indicated by the relative twist of the line joining the ends of the axis of symmetry of a uni-axial cross-section; the true associated torsion is given by the relative twist indicated by the line joining points on the axis of symmetry equidistant from the centroid.

It would appear from this analysis that Duncan, Ellis and Scruton's experiment determines the position of the point on the axis of symmetry of uni-axial cross-sections at which the load must be placed if there is to be no so-called "total torsion". But this is not the flexural centre proper, and again it seems preferable to retain our definition of the flexural centre, which is not confined to uni-axial cross-sections, rather than modify the definition to include the misnamed "anticlastic torsion" for uni-axial cross-sections.

14. Boundary problems in complex co-ordinates

In boundary problems of the type of the torsion problem, where a harmonic function is sought which takes up certain values over the boundary of the cross-section, the following method, apparently new,* sometimes proves to be the most simple and speedy way of discovering the function. We use conjugate complex variables

$$z = x+iy, \quad \bar{z} = x-iy, \tag{14·1}$$

it being unlikely that this use of z will be confused in the sections of the paper which follow with the distance of a cross-section from the root end.

We have to determine the form of a complex function ω, or real conjugate functions ϕ, ψ given by

$$\omega = \phi+i\psi = f(z), \tag{14·2}$$

such that ϕ and ψ, together with their derivatives, are finite and continuous across the cross-section bounded by the curve whose equation is

$$h(z, \bar{z}) = 0 \tag{14·3}$$

and such that
$$\psi = \psi_0(z, \bar{z}) \tag{14·4}$$
on this boundary.

Since from (14·2) we have $\psi = -i[f(z)-\bar{f}(\bar{z})]/2$ (14·5)

(the bar over f is to remind us that any i occurring in the functional form is to be changed to $-i$), the problem is to discover $f(z)$ so that on the boundary the right-hand sides of (14·4) and (14·5) are identical. The method we adopt is to use, when possible, the equation (14·3) of the boundary to express the right-hand side of equation (14·4) in a form separable in z and $\bar{z}$, and so identify $f(z)$ at once. If the resulting solution for ϕ and ψ satisfies the conditions of finiteness and continuity across the cross-section, then it follows from well-known results in two-dimensional potential theory that the solution is uniquely determined.

Consider in this manner the torsion problem. Here the boundary condition is

$$\psi = z\bar{z}/2+\text{const.} \tag{14·6}$$

Consider any closed boundary whose equation is

$$z\bar{z} = \Sigma(a_n z^n+\bar{a}_n \bar{z}^n), \tag{14·7}$$

where the a_n's are complex constants. If we write $-if(z) = \Sigma a_n z^n$ so that $i\bar{f}(\bar{z}) = \Sigma \bar{a}_n \bar{z}^n$, the resulting value of ψ in (14·5) satisfies the boundary equation (14·6) in virtue of (14·7), and

$$\omega = f(z) = i\Sigma a_n z^n \tag{14·8}$$

* Since this was written, the advantages of consistently using the complex variable in similar boundary problems of mathematical physics have been amply and more generally demonstrated by Miss Rosa M. Morris (1937 *a*, *b*, *c*, *d*, 1938).

is the solution required, provided it remains finite and continuous across the cross-section. A large number of Saint-Venant's simple torsion solutions can obviously be obtained in this way.

Since $$x^2/a^2+y^2/b^2 = 1$$

can be rewritten as $$z\bar{z} = \frac{2a^2b^2}{a^2+b^2}+\frac{1}{2}\frac{a^2-b^2}{a^2+b^2}(z^2+\bar{z}^2),$$

we have at once $$\omega = \frac{i}{2}\frac{a^2-b^2}{a^2+b^2}z^2 \tag{14·9}$$

for the elliptic cross-section.

Again the equilateral triangle of height $3h$, the origin of co-ordinates being at the centroid, has the boundary whose equation is

$$(x-h)\{(x+2h)^2-3y^2\} = 0,$$

which is equivalent to $$z\bar{z} = 4h^2/3-(z^3+\bar{z}^3)/6h,$$

so that $$\omega = -iz^3/6h \tag{14·10}$$

for the equilateral triangular cross-section.

That Saint-Venant did not exhaust the possibilities here is shown by a very interesting example due to Weber (1921, p. 31; cf. Timoshenko 1934, p. 238). A circular cylinder of radius a, with a notch whose boundary is a circle of radius b, and with its centre on the circumference of the cylinder, has for its equation

$$(x^2+y^2-b^2)[(x-a)^2+y^2-a^2] = 0$$

or $$z\bar{z} = b^2+a(z+\bar{z})-ab^2(1/z+1/\bar{z}),$$

so that $$\omega = iaz-iab^2/z, \tag{14·11}$$

since negative powers can be admitted in (14·8) if the origin is outside the cross-section, as it is in this case.

The torsion moment in these cases is then readily calculated via equations (7·4) and (8·4).

An interesting example of the use of this method occurs when two pairs of real conjugate harmonic functions have to be determined to satisfy the boundary condition

$$\psi_1+i\psi_2 = g(z, \bar{z}) \tag{14·12}$$

round the boundary of equation (14·3). If in virtue of (14·3) it is possible to rewrite (14·12) in the separable form

$$\psi_1+i\psi_2 = f(z)+F(\bar{z}), \tag{14·13}$$

then, along the boundary,

$$2\psi_1 = f(z)+\bar{F}(z)+\bar{f}(\bar{z})+F(\bar{z}),$$

and $$2i\psi_2 = f(z)-\bar{F}(z)+F(\bar{z})-\bar{f}(\bar{z}).$$

But if $$\omega_1 = \phi_1+i\psi_1 = f_1(z), \quad \omega_2 = \phi_2+i\psi_2 = f_2(z)$$

throughout the cross-section, we have

$$2i\psi_1 = f_1(z) - \bar{f}_1(\bar{z}), \quad 2i\psi_2 = f_2(z) - \bar{f}_2(\bar{z});$$

hence if we take $\quad -if_1(z) = f(z) + \bar{F}(z), \quad f_2(z) = f(z) - \bar{F}(z),$

so that necessarily $\quad i\bar{f}_1(\bar{z}) = \bar{f}(\bar{z}) + F(\bar{z}), \quad \bar{f}_2(\bar{z}) = \bar{f}(\bar{z}) - F(\bar{z}),$

then the solutions
$$\omega_1 = \phi_1 + i\psi_1 = i\{f(z) + \bar{F}(z)\}, \tag{14·14}$$

$$\omega_2 = \phi_2 + i\psi_2 = f(z) - \bar{F}(z) \tag{14·15}$$

satisfy the boundary conditions for the functions ψ_1, ψ_2 and, provided their form satisfies the physical requirements of the problem as to finiteness and continuity across the cross-section, they will be the solutions holding throughout the cross-section.

Obviously the determination of the canonical flexure functions ϕ_1, ϕ_2 and their conjugates ψ_1, ψ_2 is a problem of this type, in which the boundary condition is

$$\psi_1 + i\psi_2 = \bar{z}z^2/4 + \bar{z}^3/12. \tag{14·16}$$

To illustrate, consider the flexure problem for the circular cross-section of boundary $z\bar{z} = a^2$, by means of which the boundary condition can be rewritten in the separable form

$$\psi_1 + i\psi_2 = a^2 z/4 + \bar{z}^3/12.$$

Comparing with (14·13), we may take

$$f(z) = a^2 z/4, \quad F(\bar{z}) = \bar{z}^3/12,$$

giving at once from (14·14) and (14·15) the solutions

$$\omega_1 = ia^2 z/4 + iz^3/12, \quad \omega_2 = a^2 z/4 - z^3/12, \tag{14·17}$$

which are readily seen to be satisfactory solutions in every way.

By a simple adaptation of the method we can find the two remaining non-zero canonical flexure functions χ_1, χ_2 for the circular cross-section, and their conjugate functions χ_1^*, χ_2^*, where

$$\Omega_1 = \chi_1 + i\chi_1^*, \quad \Omega_2 = \chi_2 + i\chi_2^*.$$

These have to satisfy the boundary condition

$$\frac{\partial}{\partial n}\{\chi_1 + i\chi_2 - \bar{z}z^2/4 - \bar{z}^3/12\} = 0$$

over the circle $z\bar{z} = a^2$. This boundary is also given by

$$\xi = \alpha = \log a,$$

where $\quad \xi + i\eta = \zeta = \log z = \log r + i\theta,$

and since $\partial/\partial\xi = \partial/\partial\zeta + \partial/\partial\bar{\zeta}$, the boundary condition can be rewritten

$$\left(\frac{\partial}{\partial\zeta}+\frac{\partial}{\partial\bar{\zeta}}\right)\{\chi_1+i\chi_2\} = \tfrac{1}{2}z\bar{z}\frac{dz}{d\zeta}+\tfrac{1}{4}(z^2+\bar{z}^2)\frac{d\bar{z}}{d\bar{\zeta}},$$

or

$$\frac{\partial}{\partial\xi}(\chi_1+i\chi_2) = 3\bar{z}z^2/4+\bar{z}^3/4,$$

and, on making use of the equation of the boundary to render the right-hand side of this equation separable in z and $\bar{z}$, we have

$$\frac{\partial}{\partial\xi}(\chi_1+i\chi_2) = 3a^2z/4+\bar{z}^3/4. \tag{14·18}$$

But if

$$\chi_1+i\chi_2 = f(z)+F(\bar{z}), \tag{14·19}$$

so that

$$\Omega_1 = f(z)+\bar{F}(z), \tag{14·20}$$

$$\Omega_2 = -i\{f(z)-\bar{F}(z)\}, \tag{14·21}$$

we have in this problem

$$\frac{\partial}{\partial\xi}(\chi_1+i\chi_2) = f'(z)\frac{dz}{d\zeta}+F'(\bar{z})\frac{d\bar{z}}{d\bar{\zeta}}, \qquad \frac{dz}{d\zeta} = z,$$

and the boundary condition (14·18) is satisfied by taking

$$f'(z)\frac{dz}{d\zeta} = 3a^2z/4+C, \quad F'(\bar{z})\frac{d\bar{z}}{d\bar{\zeta}} = \bar{z}^3/4-C.$$

Hence

$$f(z) = 3a^2z/4 \quad \text{and} \quad F(\bar{z}) = \bar{z}^3/12,$$

the terms of the solutions arising from the constant C being rejected as they become infinite at the origin. Accordingly from (14·20) and (14·21) we find

$$\Omega_1 = 3a^2z/4+z^3/12, \quad \Omega_2 = -i\{3a^2z/4-z^3/12\}, \tag{14·22}$$

which completes the discovery of the canonical flexure functions in this simple case. Several further examples of the use of the methods given in this section will occur in the later sections.

In the cases above the physical suitability of the canonical flexure functions has been fairly obvious: we shall discover that in the general case it is not always possible to avoid singularities in the functions χ_1, χ_2, χ_3 considered separately, and we may have to include infinities at certain points. But from the nature of the physical problem, in which our functions always occur in the combinations $\chi_1-h\chi_3$, $\chi_2-k\chi_3$, these combinations must almost always be free of such infinities, infinite stresses at a re-entrant angle being the only allowable possibility. The avoidance of unphysical solutions in the combinations, when they occur in the separate canonical functions, affords a check on the calculated values of the co-ordinates (h, k) of the centroid of the cross-section. This feature cannot occur with the three remaining canonical functions, since ψ_3 must give a physical solution separately, being the torsion function, whereas χ_3 separately *need* not be a physically admissible solution in the sense explained above.

15. Cross-section a circular sector

The next three sections are devoted to the particular solutions for certain cross-sections (for which the classical Saint-Venant flexure functions are known) with the various objects of clearing up several puzzling discrepancies of signs of the associated flexural twists, exhibiting the advantages of the canonical flexure functions, and illustrating the method of § 14 where convenient.

For our first example we take the boundary to be given, using polar co-ordinates (r, θ) such that $z = x+iy = re^{i\theta}$, by

$$\left.\begin{aligned} r &= a, \quad -\beta<\theta<\beta, \\ \theta &= \pm\beta, \quad 0<r<a, \end{aligned}\right\} \tag{15·1}$$

as indicated in fig. 3.

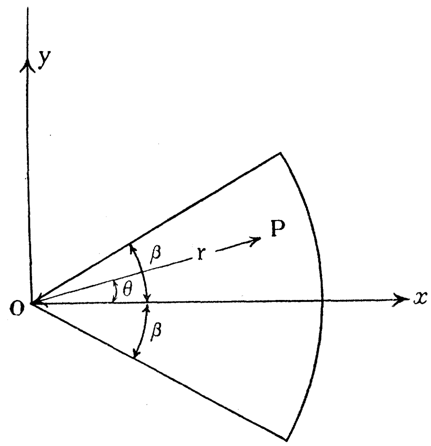

Fig. 3. Circular sector cross-section and scheme of co-ordinates.

The canonical flexure functions

The canonical flexure function ψ_1 satisfies the boundary condition

$$\psi_1 = \frac{r^3}{3}\cos^3\theta + \text{const.} = \frac{r^3}{12}(\cos 3\theta + 3\cos\theta) + \text{const.},$$

which becomes $$\psi_1 = \frac{r^3}{12}(\cos 3\beta + 3\cos\beta) \quad \text{along } \theta = \pm\beta$$

$$= \frac{a^3}{12}(\cos 3\theta + 3\cos\theta) \quad \text{along } r = a.$$

Accordingly we take the plane harmonic solution

$$\psi_1 = Ar^3\cos 3\theta + \Sigma a^3 A_m (r/a)^m \cos m\theta,$$

which satisfies the boundary condition along $\theta = \pm\beta$, if

$$\cos m\beta = 0, \quad \text{i.e. } m\beta = (n+\tfrac{1}{2})\,\pi, \quad n = 0, 1, 2, 3, \ldots, \tag{15·2}$$

and $$A = (\cos 3\beta + 3\cos\beta)/12\cos 3\beta, \tag{15·3}$$

and will also satisfy the remaining boundary condition along $r=a$, if

$$\sum_{n=0}^{\infty} A_n \cos m\theta = -\frac{1}{4}\left\{\frac{\cos\beta}{\cos 3\beta}\cos 3\theta - \cos\theta\right\} = f(\theta), \quad -\beta<\theta<\beta,$$

replacing A_m by A_n.

From the Fourier expansion as a cosine series for the range $0\leqslant\theta\leqslant 2\beta$ of the function $F(\theta)$, given by

$$\left.\begin{aligned} F(\theta) &= f(\theta), \quad \text{for } 0\leqslant\theta\leqslant\beta, \\ &= -f(\theta-2\beta), \quad \text{for } \beta\leqslant\theta\leqslant 2\beta, \end{aligned}\right\} \tag{15·4}$$

we find
$$A_n = (-1)^n \frac{\cos\beta}{4\beta}\left\{\frac{1}{m-1}+\frac{1}{m+1}-\frac{1}{m-3}-\frac{1}{m+3}\right\}. \tag{15·5}$$

Hence
$$\omega_1 = \phi_1 + i\psi_1 = iAz^3 + ia^3 \sum_{n=0}^{\infty} A_n (z/a)^m, \tag{15·6}$$

where m, A and A_n are given by (15·2), (15·3) and (15·5) respectively.

Next consider the canonical flexure function ψ_2 which has to satisfy the boundary condition

$$\psi_2 = \frac{r^3}{3}\sin^3\theta + \text{const.} = \frac{r^3}{12}(3\sin\theta - \sin 3\theta) + \text{const.},$$

and this becomes

$$\psi_2 = \pm\frac{r^3}{12}(3\sin\beta - \sin 3\beta) + \text{const.} \quad \text{along } \theta = \pm\beta,$$

$$= \frac{a^3}{12}(3\sin\theta - \sin 3\theta) + \text{const.} \quad \text{along } r = a.$$

Accordingly we write $\psi_2 = Br^3 \sin 3\theta + \Sigma a^3 B_m (r/a)^m \sin m\theta$,

which is a plane harmonic function, satisfying the boundary condition along $\theta = \pm\beta$, if

$$\sin m\beta = 0, \quad \text{i.e. } m\beta = n\pi, \quad n = 0, 1, 2, 3, \ldots, \tag{15·7}$$

and
$$B = (3\sin\beta - \sin 3\beta)/12\sin 3\beta, \tag{15·8}$$

the remaining boundary condition along $r=a$ becoming

$$\sum_{n=0}^{\infty} B_n \sin m\theta = \frac{1}{4}\left\{\sin\theta - \frac{\sin\beta}{\sin 3\beta}\sin 3\theta\right\} = f(\theta)$$

replacing B_m by B_n.

With this value of $f(\theta)$ the Fourier expansion of $F(\theta)$, defined by equations (15·4), as a sine series gives

$$B_n = (-1)^n \frac{\sin\beta}{4\beta}\left\{\frac{1}{m-3}+\frac{1}{m+3}-\frac{1}{m-1}-\frac{1}{m+1}\right\}. \tag{15·9}$$

Hence
$$\omega_2 = \phi_2 + i\psi_2 = Bz^3 + a^3 \sum_{n=0}^{\infty} B_n (z/a)^m, \tag{15·10}$$

where m, B and B_n are given by (15·7), (15·8) and (15·9) respectively.

The canonical function ψ_3 must satisfy the boundary condition

$$\psi_3 = \tfrac{1}{2}r^2 + \text{const.},$$

so that if we write
$$\psi_3 = \tfrac{1}{2}r^2 \frac{\cos 2\theta}{\cos 2\beta} + \Sigma a^2 C_m (r/a)^m \cos m\theta,$$

where m is given by (15·2), the boundary condition is satisfied along $\theta = \pm\beta$, and the remaining boundary condition along $r = a$ becomes

$$\sum_{n=0}^{\infty} C_n \cos m\theta = \frac{1}{2}\left(1 - \frac{\cos 2\theta}{\cos 2\beta}\right) = f(\theta),$$

replacing C_m by C_n.

With this value of $f(\theta)$, the Fourier expansion of $F(\theta)$, defined by equations (15·4), as a cosine series gives

$$C_n = \frac{(-1)^n}{2\beta}\left\{\frac{2}{m} - \frac{1}{m-2} - \frac{1}{m+2}\right\}. \tag{15·11}$$

Hence
$$\omega_3 = \phi_3 + i\psi_3 = \frac{i}{2}\frac{z^2}{\cos 2\beta} + ia^2 \sum_{n=0}^{\infty} C_n (z/a)^m, \tag{15·12}$$

where m, C_n are given by (15·2) and (15·11) respectively.

The canonical flexure function χ_1 must satisfy the boundary condition

$$\frac{\partial}{\partial n}\left(\chi_1 - \frac{r^3}{3}\cos^3\theta\right) = 0,$$

which becomes
$$\frac{\partial \chi_1}{\partial r} = \frac{a^2}{4}(\cos 3\theta + 3\cos\theta) \quad \text{along } r = a,$$

and
$$\frac{\partial \chi_1}{\partial \theta} = \mp \frac{r^3}{4}(\sin\beta + \sin 3\beta) \quad \text{along } \theta = \pm\beta.$$

We take therefore the plane harmonic solution

$$\chi_1 = a^3 D_0 \log r + Dr^3 \cos 3\theta + \Sigma a^3 D_m (r/a)^m \cos m\theta$$

which satisfies the boundary conditions along $\theta = \pm\beta$, if m is given by (15·7) and

$$D = (\sin\beta + \sin 3\beta)/12 \sin 3\beta. \tag{15·13}$$

The remaining boundary condition along $r = a$ becomes

$$D_0 + \sum_{n=0}^{\infty} mD_n \cos m\theta = \frac{1}{4}\left\{3\cos\theta - \frac{\sin\beta}{\sin 3\beta}\cos 3\theta\right\} = f(\theta)$$

on putting D_n for D_m.

Using this value of $f(\theta)$, the Fourier expansion of $F(\theta)$, defined in (15·4), as a cosine series leads to

$$D_0 = 2\sin\beta/3\beta, \tag{15·14}$$

$$D_n = (-1)^n \frac{\sin\beta}{12\beta}\left\{\frac{16}{m} + \frac{1}{m-3} + \frac{1}{m+3} - \frac{9}{m-1} - \frac{9}{m+1}\right\}. \tag{15·15}$$

Hence
$$\Omega_1 = \chi_1 + i\chi_1^* = a^3 D_0 \log z + Dz^3 + a^3 \sum_{n=0}^{\infty} D_m (z/a)^m, \tag{15·16}$$

where m, D, D_0, D_n are given by (15·7), (15·13), (15·14) and (15·15) respectively. The term in D_0 appears to make this an unsuitable solution since the logarithmic infinity occurs in χ_1 at the origin, but we shall show that the logarithmic infinity disappears from the combination $\chi_1 - h\chi_3$, and χ_1 and χ_3 always occur in this combination in the physical problem.

Next the canonical flexure function χ_2 must satisfy the boundary condition

$$\frac{\partial}{\partial n}\left(\chi_2 - \frac{r^3}{3}\sin^3\theta\right) = 0,$$

which becomes $$\frac{\partial \chi_2}{\partial r} = \frac{a^2}{4}(3\sin\theta - \sin 3\theta) \quad \text{along } r = a,$$

and $$\frac{\partial \chi_2}{\partial \theta} = \frac{r^3}{4}(\cos\beta - \cos 3\beta) \quad \text{along } \theta = \pm\beta.$$

Accordingly we take the plane harmonic solution

$$\chi_2 = Er^3 \sin 3\theta + \Sigma a^3 E_m (r/a)^m \sin m\theta$$

which satisfies the boundary conditions along $\theta = \pm\beta$, if m is given by (15·2) and

$$E = (\cos\beta - \cos 3\beta)/12\cos 3\beta. \tag{15·17}$$

Putting $E_m = E_n$, the remaining boundary condition along $r = a$ becomes

$$\sum_{n=0}^{\infty} mE_n \sin m\theta = \frac{1}{4}\left\{3\sin\theta - \frac{\cos\beta}{\cos 3\beta}\sin 3\theta\right\} = f(\theta).$$

With this value of $f(\theta)$, the Fourier expansion of $F(\theta)$, defined in (15·4), as a sine series gives

$$E_n = (-1)^{n+1}\frac{\cos\beta}{12\beta}\left\{\frac{16}{m} + \frac{1}{m-3} + \frac{1}{m+3} - \frac{9}{m-1} - \frac{9}{m+1}\right\}. \tag{15·18}$$

Hence $$\Omega_2 = \chi_2 + i\chi_2^* = -iEz^3 - ia^3 \sum_{n=0}^{\infty} E_n (z/a)^m, \tag{15·19}$$

where m, E and E_n are given by (15·2), (15·17) and (15·18) respectively.

Finally the remaining canonical flexure function χ_3 must satisfy the boundary condition

$$\frac{\partial}{\partial n}(\chi_3 - \tfrac{1}{2}r^2) = 0,$$

or $$\frac{\partial \chi_3}{\partial r} = a \quad \text{along } r = a, \qquad \frac{\partial \chi_3}{\partial \theta} = 0 \quad \text{along } \theta = \pm\beta.$$

These are satisfied by the solution

$$\Omega_3 = \chi_3 + i\chi_3^* = a^2 \log z. \tag{15·20}$$

The critical term in the combination $\chi_1 - h\chi_3$ is accordingly

$$a^2\left(h - \frac{2a}{3}\frac{\sin\beta}{\beta}\right)\log r$$

which vanishes since $$h = 2a\sin\beta/3\beta, \tag{15·21}$$

and so the solutions for χ_1 and χ_3 are completely satisfactory.

The moment integrals

We have the following results for this cross-section:

$$\left.\begin{aligned} S = \beta a^2, \quad \int x^2\,dS = a^4(2\beta + \sin 2\beta)/8, \quad \int xy^2\,dS = 2a^5\sin^3\beta/15, \\ \int y^2\,dS = a^4(2\beta - \sin 2\beta)/8, \quad \int x^3\,dS = 2a^5(3\sin\beta - \sin^3\beta)/15, \end{aligned}\right\}. \tag{15·22}$$

and we have to calculate the three moment integrals L_2, M_1, M_3.

We write $\Omega_2 = \Omega_{21} + \Omega_{22}$, with $L'_2 = L'_{21} + L'_{22}$ to correspond,

where $$\Omega_{21} = -iEz^3, \quad \Omega_{22} = -a^3\sum_{n=0}^{\infty} E_n\omega_n,$$

and $$\omega_n = i(z/a)^m, \quad m\beta = (n+\tfrac{1}{2})\,\pi. \tag{15·23}$$

Then from (8·3) $$L'_{21} = R\left\{3E\int z^3\,dS\right\} = 3E\int(x^3 - 3xy^2)\,dS$$

or $$L'_{21} = a^5\tan 3\beta(\cos\beta - \cos 3\beta)/30,$$

using (15·17) and (15·22).

Also $$L'_{22} = -a^3\sum_{n=0}^{\infty} E_n L'_n,$$

where $$L'_n = -R\left\{m\int(z/a)^m\,dS\right\} = (-1)^{n+1}\,2a^2/(m+2). \tag{15·24}$$

Hence from (15·18) we find

$$L'_{22} = \frac{a^5\cos\beta}{\beta}\sum_{n=0}^{\infty}\left\{-\frac{1}{30(m-3)} + \frac{1}{2(m-1)} - \frac{4}{3m} + \frac{3}{2(m+1)} - \frac{4}{5(m+2)} + \frac{1}{6(m+3)}\right\}$$

or $$L'_{22} = \frac{a^5\cos\beta}{\pi}\sum_{n=0}^{\infty}\left\{-\frac{1}{30\left(n+\frac{1}{2}-\frac{3\beta}{\pi}\right)} + \frac{1}{2\left(n+\frac{1}{2}-\frac{\beta}{\pi}\right)} - \frac{4}{3\left(n+\frac{1}{2}\right)} + \frac{3}{2\left(n+\frac{1}{2}+\frac{\beta}{\pi}\right)} - \frac{4}{5\left(n+\frac{1}{2}+\frac{2\beta}{\pi}\right)} + \frac{1}{6\left(n+\frac{1}{2}+\frac{3\beta}{\pi}\right)}\right\}.$$

We now make use of the Psi function (or digamma function) (Davis 1933, **1**, 277; Bromwich 1908, pp. 475–6; Adams 1922, p. 132) defined as

$$\psi(x) = -\gamma - \sum_{n=0}^{\infty}\left\{\frac{1}{n+x} - \frac{1}{n+1}\right\}, \tag{15·25}$$

where γ is Euler's constant, and so obtain

$$L'_{22} = \frac{a^5}{\pi}\cos\beta\left\{-\tfrac{3}{2}\psi\left(\frac{1}{2}+\frac{\beta}{\pi}\right)+\tfrac{4}{5}\psi\left(\frac{1}{2}+\frac{2\beta}{\pi}\right)-\tfrac{1}{6}\psi\left(\frac{1}{2}+\frac{3\beta}{\pi}\right)+\tfrac{4}{3}\psi(\tfrac{1}{2})\right.$$
$$\left.+\tfrac{1}{30}\psi\left(\frac{1}{2}-\frac{3\beta}{\pi}\right)-\tfrac{1}{2}\psi\left(\frac{1}{2}-\frac{\beta}{\pi}\right)\right\}.$$

But the digamma function satisfies the equation

$$\psi(1-x) = \psi(x)+\pi\cot\pi x; \tag{15·26}$$

hence

$$\psi\left(\frac{1}{2}-\frac{3\beta}{\pi}\right) = \psi\left(\frac{1}{2}+\frac{3\beta}{\pi}\right)-\pi\tan 3\beta,$$

$$\psi\left(\frac{1}{2}-\frac{\beta}{\pi}\right) = \psi\left(\frac{1}{2}+\frac{\beta}{\pi}\right)-\pi\tan\beta,$$

whence, on collecting up our results,

$$L'_2 = \tfrac{1}{2}a^5\sin\beta-\tfrac{1}{30}a^5\sin 3\beta$$
$$-(2a^5/\pi)\cos\beta\left\{\psi\left(\frac{1}{2}+\frac{\beta}{\pi}\right)-\tfrac{2}{5}\psi\left(\frac{1}{2}+\frac{2\beta}{\pi}\right)+\tfrac{1}{15}\psi\left(\frac{1}{2}+\frac{3\beta}{\pi}\right)-\tfrac{2}{3}\psi(\tfrac{1}{2})\right\}.$$

Finally, from (7·4) and (15·22), we have

$$L_2 = \tfrac{2}{5}a^5\sin\beta-(2a^5/\pi)\cos\beta\left\{\psi\left(\frac{1}{2}+\frac{\beta}{\pi}\right)-\tfrac{2}{5}\psi\left(\frac{1}{2}+\frac{2\beta}{\pi}\right)+\tfrac{1}{15}\psi\left(\frac{1}{2}+\frac{3\beta}{\pi}\right)-\tfrac{2}{3}\psi(\tfrac{1}{2})\right\}. \tag{15·27}$$

Next, from (15·6) we have

$$\omega_1 = \omega_{11}+\omega_{12}, \quad \text{with } M'_1 = M'_{11}+M'_{12} \text{ to correspond,}$$

where

$$\omega_{11} = iAz^3, \quad \omega_{12} = a^3\sum_{n=0}^{\infty}A_n\omega_n$$

using (15·23). Then, from (8·4),

$$M'_{11} = R\left\{-3A\int z^3\,dS\right\} = -3A\int(x^3-3xy^2)\,dS$$

or

$$M'_{11} = -a^5\tan 3\beta(3\cos\beta+\cos 3\beta)/30$$

on making use of (15·3) and (15·22).

Also

$$M'_{12} = a^3\sum_{n=0}^{\infty}A_nL'_n,$$

and using (15·5) and (15·24) we have

$$M'_{12} = \frac{a^5\cos\beta}{\beta}\sum_{n=0}^{\infty}\left\{\frac{1}{10(m-3)}-\frac{1}{6(m-1)}-\frac{1}{2(m+1)}+\frac{16}{15(m+2)}+\frac{1}{2(m+3)}\right\},$$

leading to

$$M'_{12} = \frac{a^5}{\pi}\cos\beta\left\{\tfrac{1}{6}\psi\left(\frac{1}{2}-\frac{\beta}{\pi}\right)+\tfrac{1}{2}\psi\left(\frac{1}{2}+\frac{\beta}{\pi}\right)-\tfrac{1}{10}\psi\left(\frac{1}{2}-\frac{3\beta}{\pi}\right)+\tfrac{1}{2}\psi\left(\frac{1}{2}+\frac{3\beta}{\pi}\right)-\tfrac{16}{15}\psi\left(\frac{1}{2}+\frac{2\beta}{\pi}\right)\right\}$$

on using (15·25), which reduces further on making use of (15·26) to

$$M'_{12} = a^5\cos\beta(\tfrac{1}{10}\tan 3\beta-\tfrac{1}{6}\tan\beta)+\frac{a^5}{\pi}\cos\beta\left\{\tfrac{2}{5}\psi\left(\frac{1}{2}+\frac{3\beta}{\pi}\right)-\tfrac{16}{15}\psi\left(\frac{1}{2}+\frac{2\beta}{\pi}\right)+\tfrac{2}{3}\psi\left(\frac{1}{2}+\frac{\beta}{\pi}\right)\right\}.$$

Hence finally from (7·4) and (15·22) these results give

$$M_1 = \tfrac{2}{15}a^5\sin\beta+\frac{a^5}{\pi}\cos\beta\left\{\tfrac{2}{5}\psi\left(\frac{1}{2}+\frac{3\beta}{\pi}\right)-\tfrac{16}{15}\psi\left(\frac{1}{2}+\frac{2\beta}{\pi}\right)+\tfrac{2}{3}\psi\left(\frac{1}{2}+\frac{\beta}{\pi}\right)\right\}. \tag{15·28}$$

In similar fashion, from (15·12) we write

$$\omega_3 = \omega_{31}+\omega_{32}, \quad \text{with } M'_3 = M'_{31}+M'_{32} \text{ to correspond,}$$

where
$$\omega_{31} = iz^2/2\cos 2\beta, \quad \omega_{32} = a^2\sum_{n=0}^{\infty} C_n\omega_n$$

using (15·23). Then from (8·4) we have

$$M'_{31} = R\left\{-\int z^2\,dS/\cos 2\beta\right\} = -\sec 2\beta\int(x^2-y^2)\,dS$$

or
$$M'_{31} = -\frac{a^4}{4}\tan 2\beta$$

from (15·22). Next we find
$$M'_{32} = a^2\sum_{n=0}^{\infty} C_n L'_n,$$

and using (15·11) and (15·24) we have

$$M'_{32} = \frac{a^4}{\beta}\sum_{n=0}^{\infty}\left\{\frac{3}{4}\frac{1}{m+2}+\frac{1}{4}\frac{1}{m-2}+\frac{1}{(m+2)^2}-\frac{1}{m}\right\}$$

or
$$M'_{32} = \frac{a^4}{\pi}\sum_{n=0}^{\infty}\left\{\frac{3}{4}\frac{1}{\left(n+\frac{1}{2}+\frac{2\beta}{\pi}\right)}+\frac{1}{4}\frac{1}{\left(n+\frac{1}{2}-\frac{2\beta}{\pi}\right)}-\frac{1}{\left(n+\frac{1}{2}\right)}+\frac{\beta}{\pi}\frac{1}{\left(n+\frac{1}{2}+\frac{2\beta}{\pi}\right)^2}\right\},$$

on using (15·23).

If now we use the trigamma function defined as

$$\psi'(x) = \sum_{n=0}^{\infty}\frac{1}{(n+x)^2}, \tag{15·29}$$

together with (15·25) and (15·26) we find

$$M'_{32} = \frac{a^4}{4}\tan 2\beta+\frac{a^4}{\pi}\left\{\psi(\tfrac{1}{2})-\psi\left(\frac{1}{2}+\frac{2\beta}{\pi}\right)+\frac{\beta}{\pi}\psi'\left(\frac{1}{2}+\frac{2\beta}{\pi}\right)\right\}.$$

TABLE I

1	2	3	4	5	6	7	8	9
β/π	$\beta°$	L_2/a^5	M_1/a^5	M_3/a^4	h/a	f_0/a	$-\tau'/(W'a/EI')$	$-\{\tau'/(W'a/EI')+\eta(\frac{1}{2}-h/a)\}$
0·00	0° 00′	0·0000000	0·0000000	0·0000000	0·6666667	0·800000 +0·266667 σ	0·066667 +0·200000 η	0·066667 +0·366667 η
0·05	9° 00′	0·0009918	0·0013283	0·0018016	0·6639285	0·771514 +0·102859 σ	0·076771 +0·150170 η	0·077677 +0·314098 η
0·10	18° 00′	0·0073201	0·0073799	0·0106311	0·6557544	0·722377 +0·040307 σ	0·063503 +0·101922 η	0·063503 +0·257676 η
0·15	27° 00′	0·0224951	0·0181974	0·0275784	0·6422652	0·674208 +0·014530 σ	0·038646 +0·056225 η	0·038646 +0·198490 η
0·20	36° 00′	0·0483086	0·0326373	0·0518480	0·6236596	0·632351 +0·003951 σ	0·012807 +0·018628 η	0·012807 +0·142287 η
0·21	37° 48′	0·0548122	0·0358483	0·0574822	0·6193473	0·624844 +0·002813 σ	0·007786 +0·012079 η	0·007786 +0·131426 η
0·22	39° 36′	0·0617643	0·0391442	0·0633557	0·6148435	0·617622 +0·001904 σ	0·004386 +0·007392 η	0·004386 +0·122235 η
0·23	41° 24′	0·0691627	0·0425153	0·0694600	0·6101511	0·610684 +0·001186 σ	0·000870 +0·002803 η	0·000870 +0·112954 η
0·24	43° 12′	0·0770037	0·0459526	0·0757865	0·6052725	0·604003 +0·000636 σ	−0·002136 −0·001066 η	−0·002136 +0·104206 η
0·25	45° 00′	0·0852825	0·0494470	0·0823276	0·6002109	0·597639 +0·000232 σ	−0·004458 −0·004056 η	−0·004458 +0·096155 η
0·30	54° 00′	0·1329886	0·0674768	0·1179896	0·5722625	0·569606 −0·000189 σ	−0·005257 −0·005632 η	−0·005257 +0·066631 η
0·31	55° 48′	0·1437264	0·0711375	0·1256653	0·5661676	0·564734 −0·000040 σ	−0·002903 −0·002983 η	−0·002903 +0·063184 η
0·32	57° 36′	0·1548366	0·0747985	0·1335073	0·5599123	0·560093 +0·000167 σ	0·000375 +0·000720 η	0·000375 +0·060632 η
0·33	59° 24′	0·1663057	0·0784523	0·1415099	0·5535004	0·555674 +0·000423 σ	0·004598 +0·005492 η	0·004598 +0·058993 η
0·34	61° 12′	0·1781201	0·0820920	0·1496675	0·5469355	0·551475 +0·000723 σ	0·009796 +0·011356 η	0·009796 +0·058291 η
0·35	63° 00′	0·1902646	0·0857108	0·1579749	0·5402213	0·547486 +0·001063 σ	0·015980 +0·018319 η	0·015980 +0·058540 η
0·40	72° 00′	0·2553717	0·1032598	0·2015866	0·5045511	0·530508 +0·003218 σ	0·061982 +0·069666 η	0·061982 +0·074217 η
0·45	81° 00′	0·3261979	0·1187327	0·2482645	0·4657644	0·518100 +0·004924 σ	0·132724 +0·145210 η	0·132724 +0·110975 η
0·50	90° 00′	0·4000000	0·1333333	0·2975568	0·4244132	0·509296 +0·008972 σ	0·224047 +0·247727 η	0·224047 +0·172141 η
0·55	99° 00′	0·4741834	0·1446509	0·3490972	0·3810799	0·503811 +0·012343 σ	0·330894 +0·364171 η	0·330894 +0·245251 η
0·60	108° 00′	0·5460338	0·1528677	0·4025866	0·3363674	0·501213 +0·016018 σ	0·446083 +0·489429 η	0·446083 +0·325797 η
0·65	117° 00′	0·6124466	0·1576557	0·4577783	0·2908884	0·500663 +0·020023 σ	0·560557 +0·614062 η	0·560557 +0·404951 η
0·70	126° 00′	0·6706969	0·1587977	0·5144676	0·2452553	0·501523 +0·024393 σ	0·666148 +0·729557 η	0·666148 +0·474812 η
0·75	135° 00′	0·7181604	0·1561878	0·5724828	0·2000703	0·502879 +0·029165 σ	0·755377 +0·828133 η	0·755377 +0·528203 η
0·80	144° 00′	0·7524756	0·1498290	0·6316790	0·1559149	0·503530 +0·034355 σ	0·822374 +0·903650 η	0·822374 +0·559565 η
0·85	153° 00′	0·7716223	0·1398298	0·6919329	0·1133409	0·501891 +0·039940 σ	0·863333 +0·952078 η	0·863333 +0·565419 η
0·90	162° 00′	0·7739906	0·1263998	0·7531389	0·0728616	0·495937 +0·045830 σ	0·876702 +0·971671 η	0·876702 +0·544532 η
0·95	171° 00′	0·7584380	0·1098414	0·8152062	0·0349436	0·483232 +0·051835 σ	0·863087 +0·962884 η	0·863087 +0·497827 η
1·00	180° 00′	0·7243318	0·0905415	0·8780557	0·0000000	0·461124 +0·057641 σ	0·824927 +0·928043 η	0·824927 +0·428043 η

Hence finally from (7·4) and (15·22) these results give

$$M_3 = \frac{\beta}{2}a^4 + \frac{a^4}{\pi}\left\{\psi(\tfrac{1}{2}) - \psi\left(\frac{1}{2}+\frac{2\beta}{\pi}\right) + \frac{\beta}{\pi}\psi'\left(\frac{1}{2}+\frac{2\beta}{\pi}\right)\right\}. \tag{15·30}$$

These results in terms of the digamma and trigamma functions appear to be new.

The associated twist and the flexural centre

The results (15·27), (15·28) and (15·30) used in (11·13) and (11·14) enable us to compute very easily the associated twist, the position of the centre of flexure and the torsion moment for the cross-section, thanks to the recent extensive and useful tables (Davis 1933, 1935) of the digamma and trigamma functions. In certain cases the results for L_2, M_1 and M_3 are expressible in simple terms. This arises from Gauss's results (Davis 1935, **1**, 286) for $\psi(p/q)$, where p and q are integers. In particular we have

$$\psi(\tfrac{1}{2}) = -2\log 2 - \gamma, \tag{15·31}$$

$$\psi(\tfrac{1}{4}) = -3\log 2 - \gamma - \pi/2, \tag{15·32}$$

and we have also the simple results

$$\psi'(1) = \pi^2/6, \quad \psi'(\tfrac{1}{2}) = \pi^2/2. \tag{15·33}$$

From these we are led to the following results:

(1) *The quadrant section.*

When the angle 2β of the sector is $\pi/2$, we find

$$\left.\begin{aligned} L_2/a^5 &= (\surd 2/15\pi)\{28\log 2 - 4\pi - 4\} = 0{\cdot}0852825,\\ M_1/a^5 &= (\surd 2/15\pi)\{12+2\pi-24\log 2\} = 0{\cdot}0494470,\\ M_3/a^4 &= (1/6\pi)\{\pi^2 - 12\log 2\} = 0{\cdot}0823276.\end{aligned}\right\} \tag{15·34}$$

(2) *The semi-circular section.*

When $2\beta = \pi$, we have

$$\left.\begin{aligned} L_2/a^5 &= \tfrac{2}{5} = 0{\cdot}4000000,\\ M_1/a^5 &= \tfrac{2}{15} = 0{\cdot}1333333,\\ M_3/a^4 &= (\pi^2-8)/2\pi = 0{\cdot}2975568.\end{aligned}\right\} \tag{15·35}$$

(3) *The circular section with quadrantal notch.*

When $2\beta = 3\pi/2$, we find

$$\left.\begin{aligned} L_2/a^5 &= (\surd 2/15\pi)\{\tfrac{1174}{21} - 28\log 2 - 4\pi\} = 0{\cdot}7181604,\\ M_1/a^5 &= (\surd 2/15\pi)\{2\pi + 24\log 2 - \tfrac{124}{7}\} = 0{\cdot}1561878,\\ M_3/a^4 &= (1/4\pi)\{2\pi^2 - 7 - 8\log 2\} = 0{\cdot}5724828.\end{aligned}\right\} \tag{15·36}$$

(4) *The circular section with complete radial slit.*

When $\beta = \pi$, we find

$$\left.\begin{aligned} L_2/a^5 &= 512/225\pi &&= 0{\cdot}7243318, \\ M_1/a^5 &= 64/225\pi &&= 0{\cdot}0905415, \\ M_3/a^4 &= (9\pi^2-64)/9\pi &&= 0{\cdot}8780557. \end{aligned}\right\} \qquad (15{\cdot}37)$$

With some labour this list could be extended. Both L_2 and M_1 can be expressed readily enough in a variety of cases using Gauss's result for $\psi(p/q)$, but Davis's analogous result (1935, **2**, 17) for the trigamma function $\psi'(p/q)$, which would be needed for M_3, has not the same simplicity. It is much more profitable to use the tables which admit ready computation in any particular case. This has been done at intervals of 9° in β (and at closer intervals in certain interesting regions where the associated twist changes sign). The results are given in Table I for the corresponding moment integrals, associated twist, co-ordinates of the centre of flexure, and the so-called "total torsion".

Comparison with previous results

It appears from Table I that the associated twist τ' in column 8 changes sign twice, the centre of flexure moves along the axis of symmetry and passes through the centroid, coinciding with it at approximately 42° and 56°, the exact values being dependent upon the elastic constant. Thus so far as the mean twist of the cross-section is concerned we do find a change of sign for similar cross-sections even among the family of circular sectors, without having to consider other cross-sections as Young, Elderton and Pearson suggested. Even if we use their "total torsion", column 9 shows that we still get this change of sign as a possibility. It is true it will only occur for small and unpractical values of the elastic constant, but it is only in this practical sense that we can endorse the statement of Young, Elderton and Pearson's (1918, p. 23) that "since the total torsion is always negative the angle edge of the prism will always move in direction of flexing force". Incidentally they should have been able to deduce this change of sign from their own table of values for their twist τ, from the value when our $\beta = 45°$, at least for small values of η, which should have shown them that there is nothing anomalous in a change of sign of the associated torsion. Allowing for differences of definition already discussed in § 13, our numerical results for τ' will be found to be consistent with Young, Elderton and Pearson's values for their twist τ. The use of the canonical flexure functions results in a tremendous simplification of the algebra of the problem, and the tabulation of the digamma and trigamma functions has brought a corresponding gain in simplification of the necessary computations. It is clear that a similar treatment of the curtate sectors would result in a similar gain in the algebra of their case, and a proper treatment of the associated twist and the moment integrals would be worth carrying out to obtain the positions of the flexural centres, which are of special importance in thin walled-sections.*

* For one case of the curtate sector (the semi-circular "gutter" section, $\beta = \pi/2$) see Leibenson (1935, p. 17).

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Table I also shows to what extent an experimental method which treats the "total torsion" as the real associated torsion can be in error. In Duncan, Ellis and Scruton's experiments on the thin triangular sections, for example, which are comparable with thin circular sector cross-sections, it would appear from the difference between the coefficients of η in columns 8 and 9 that the influence of η on the value of the associated twist is erroneously approximately doubled.

Timoshenko (1934, p. 301) obtains a solution for the semi-circular cross-section, giving the abscissa of the flexural centre as

$$f_0/a = (8/15\pi)\,(3+4\eta)/(1+\eta) = (8/15\pi)\,(4-\sigma). \tag{15·38}$$

But equations (11·14) and (15·35) lead to

$$f_0/a = (8/15\pi)\,\{3+4\sigma(10/\pi^2-1)\}, \tag{15·39}$$

which does not agree with Timoshenko's value. Noticing that the solution for a circular cross-section gives no stresses across the diameter parallel to the load, Timoshenko assumes that the same displacements and stresses give the solution for the semi-circular section under half the load on the complete circular section, but acting at some point on the axis of symmetry, which he proceeds to find with the above result. All this is quite satisfactory until he calls this point the "flexural centre", which however it is not. It is the load-point when the local twist vanishes *at the midpoint of the straight boundary*, and not the load-point when the local twist vanishes at the centroid of the section, since the solution was derived from the displacements and stresses which made the local twist vanish at the centroid of the complete circular section.

This example serves to emphasize the present writer's contention in § 13 that the associated twist and the anticlastic displacements should be defined with regard to a definite point of the cross-section, otherwise such terms as "the associated twist" and "the flexural centre" can have no uniqueness of physical meaning. When the demands of the general asymmetric cross-section are taken into account, it is clear that this origin of reference should be the centroid of the cross-section, as we have consistently used.

For a thin blade section, symmetrical about the axis of y, Griffith and Taylor (1921, p. 968) find a formula for the distance $\bar{y}$ of the centre of flexure from the origin of co-ordinates (which is the centroid of the cross-section) as

$$\bar{y} = \int t^3 y\,dy \bigg/ \int t^3\,dy,$$

where $2t$ is the small thickness at a distance y from the origin. This result follows from their approximate theory of thin sections. They show (1921, p. 960) that for a circular sector of small angle and radius a, this gives $\bar{y} = 0{\cdot}8a$. Comparison with the first entry of column 7 in Table I shows that this would agree with the exact solution if the elastic constant σ were zero. It is clear from the formula that the approximate solution makes the centre of flexure independent of the elastic constant, which is not the case in the exact solution. It is just in these cases of thin sections that the influence of the elastic constant on the position of the centre of flexure becomes relatively important, as column 7 of Table I shows, for example. The approximate theory for thin sections has also been discussed by Duncan (1932, p. 890) who gives a formula which, in our notation (see note in §2), is

$$\bar{y} = (1+2\sigma)\int yt^3 dy \Big/ \int t^3 dy$$

and the result for the circular sector of small angle and radius a is then $(0{\cdot}8+1{\cdot}6\sigma)a$. For values of σ greater than 0·125 this brings the centre of flexure outside the cross-section. Duncan's formula would here appear to overestimate considerably the influence of the elastic constant.

Using the method of generalized plane stress in his work on the behaviour in bending of thin-walled tubes and channels, Williams (1935) also finds a formula for the position of a so-called centre of flexure which is independent of the elastic constant. This centre should, however, be distinguished as the "centre of least strain". These approximate methods need further comparison with exact solutions.

16. Cross-section a cardioid

This section illustrates well the method of § 14 for a uni-axial cross-section, the solution for which, given by Shepherd by a less direct process, was the first solution in finite terms for a prism of uni-axial symmetry.

Using orthogonal curvilinear co-ordinates defined by

$$\xi + i\eta = \zeta = z^{-\frac{1}{2}}, \quad \text{where } z = x+iy = re^{i\theta}, \tag{16·1}$$

we find

$$\xi = r^{-\frac{1}{2}}\cos\frac{\theta}{2}, \quad \eta = r^{-\frac{1}{2}}\sin\frac{\theta}{2}, \tag{16·2}$$

so that $\xi=$ const., $\eta=$ const. give two families of orthogonal cardioids. We shall take as the boundary of the cylinder

$$\xi = \alpha = (2a)^{-\frac{1}{2}}, \quad \text{or} \quad r = a(1+\cos\theta), \tag{16·3}$$

and the equation of the boundary can also be written

$$\zeta + \bar{\zeta} = 2\alpha, \quad \text{or} \quad z^{-\frac{1}{2}} + \bar{z}^{-\frac{1}{2}} = 2\alpha, \tag{16·4}$$

whence

$$(z\bar{z})^{\frac{1}{2}} = (z^{\frac{1}{2}} + \bar{z}^{\frac{1}{2}})/2\alpha \tag{16·5}$$

and

$$z\bar{z} = (z+\bar{z})/4\alpha^2 + (z^{\frac{1}{2}} + \bar{z}^{\frac{1}{2}})/4\alpha^3. \tag{16·6}$$

The canonical flexure functions

Using (16·6), we have by inspection on comparison with (14·6) and (14·7)

$$\omega_3 = \phi_3 + i\psi_3 = iz/4\alpha^2 + iz^{\frac{1}{2}}/4\alpha^3 = iaz/2 + i2^{\frac{1}{2}}a^{\frac{3}{2}}z^{\frac{1}{2}}/2, \tag{16·7}$$

substituting for α from (16·3).

Again $\bar{z}z^2 = z(z\bar{z})$, and using (16·5), (16·6), we have

$$\bar{z}^3/12 + \bar{z}z^2/4 = \bar{z}^3/12 + z^2/16\alpha^2 + z^{\frac{3}{2}}/16\alpha^3 + (3z+\bar{z})/64\alpha^4 + (z^{\frac{1}{2}} + \bar{z}^{\frac{1}{2}})/32\alpha^5. \tag{16·8}$$

Hence the boundary condition for ψ_1 and ψ_2,

$$\psi_1 + i\psi_2 = \bar{z}^3/12 + \bar{z}z^2/4 = f(z) + F(\bar{z}),$$

gives

$$f(z) = z^2/16\alpha^2 + z^{\frac{3}{2}}/16\alpha^3 + 3z/64\alpha^4 + z^{\frac{1}{2}}/32\alpha^5,$$

$$F(z) = z^3/12 + z/64\alpha^4 + z^{\frac{1}{2}}/32\alpha^5,$$

and then (14·14) and (14·15) give, substituting for α from (16·3),

$$\omega_1 = \phi_1 + i\psi_1 = i\{z^3/12 + az^2/8 + 2^{\frac{1}{2}}a^{\frac{3}{2}}z^{\frac{3}{2}}/8 + a^2z/4 + 2^{\frac{1}{2}}a^{\frac{5}{2}}z^{\frac{1}{2}}/4\}, \tag{16·9}$$

$$\omega_2 = \phi_2 + i\psi_2 = -z^3/12 + az^2/8 + 2^{\frac{1}{2}}a^{\frac{3}{2}}z^{\frac{3}{2}}/8 + a^2z/8. \tag{16·10}$$

The canonical flexure functions χ_1 and χ_2 satisfy the boundary conditions

$$\frac{\partial}{\partial\xi}\{\chi_1 + i\chi_2 - \bar{z}^3/12 - \bar{z}z^2/4\} = 0,$$

$$\frac{\partial}{\partial\xi}\{\chi_1 + i\chi_2\} = \tfrac{1}{4}(z^2+\bar{z}^2)\frac{d\bar{z}}{d\bar{\zeta}} + \tfrac{1}{2}z\bar{z}\frac{dz}{d\zeta},$$

since $\partial/\partial\xi = \partial/\partial\zeta + \partial/\partial\bar{\zeta}$, or using (16·1)

$$-\frac{\partial}{\partial\xi}\{\chi_1 + i\chi_2\} = \tfrac{1}{2}(z^2+\bar{z}^2)\bar{z}^{\frac{3}{2}} + \bar{z}z^{\frac{5}{2}}.$$

But using (16·5), (16·6) this can be written in a separable form as

$$-\frac{\partial}{\partial\xi}\{\chi_1 + i\chi_2\} = \bar{z}^{\frac{7}{2}}/2 + z^{\frac{5}{2}}/4\alpha^2 + 5z^2/16\alpha^3 + (9z^{\frac{3}{2}} + \bar{z}^{\frac{3}{2}})/32\alpha^4$$
$$+ (7z+3\bar{z})/32\alpha^5 + 5(z^{\frac{1}{2}} + \bar{z}^{\frac{1}{2}})/32\alpha^6,$$

and from (14·19)

$$-f'(z)\frac{dz}{d\zeta} = z^{\frac{5}{2}}/4\alpha^2 + 5z^2/16\alpha^3 + 9z^{\frac{3}{2}}/32\alpha^4 + 7z/32\alpha^5 + 5z^{\frac{1}{2}}/32\alpha^6 + C,$$

$$-F'(\bar{z})\frac{d\bar{z}}{d\bar{\zeta}} = \bar{z}^{\frac{7}{2}}/2 + \bar{z}^{\frac{3}{2}}/32\alpha^4 + 3\bar{z}/32\alpha^5 + 5\bar{z}^{\frac{1}{2}}/32\alpha^6 - C,$$

where C is a constant, which however we take to be zero to avoid infinite displacements at the origin, and using (16·1) we have

$$f(z) = z^2/16\alpha^2 + 5z^{\frac{3}{2}}/48\alpha^3 + 9z/64\alpha^4 + 7z^{\frac{1}{2}}/32\alpha^5 + (5\log z)/64\alpha^6,$$

$$F(z) = z^3/12 + z/64\alpha^4 + 3z^{\frac{1}{2}}/32\alpha^5 + (5\log z)/64\alpha^6.$$

Hence (14·20) and (14·21) give, on substituting for α from (16·3),

$$\Omega_1 = \chi_1 + i\chi_1^* = z^3/12 + az^2/8 + 5\,.\,2^{\frac{1}{2}}a^{\frac{3}{2}}z^{\frac{3}{2}}/24 + 5a^2z/8 + 5\,.\,2^{\frac{1}{2}}a^{\frac{5}{2}}z^{\frac{1}{2}}/4 + (5a^2\log z)/4, \quad (16·11)$$

$$\Omega_2 = \chi_2 + i\chi_2^* = i\{z^3/12 - az^2/8 - 5\,.\,2^{\frac{1}{2}}a^{\frac{3}{2}}z^{\frac{3}{2}}/24 - a^2z/2 - 2^{-\frac{1}{2}}a^{\frac{5}{2}}z^{\frac{1}{2}}\}, \quad (16·12)$$

where, for (16·11) to be a physically admissible solution, we shall have to show that $\chi_1 - h\chi_3$ is independent of the logarithmic term which becomes infinite at the origin.

The remaining canonical flexure function χ_3 has to satisfy the boundary condition

$$\frac{\partial\chi_3}{\partial\xi} = \frac{\partial}{\partial\xi}(\tfrac{1}{2}z\bar{z}) = \tfrac{1}{2}z\frac{d\bar{z}}{d\zeta} + \tfrac{1}{2}\bar{z}\frac{dz}{d\zeta} = -z\bar{z}(z^{\frac{1}{2}} + \bar{z}^{\frac{1}{2}}),$$

or
$$\frac{\partial\chi_3}{\partial\xi} = -(z^{\frac{3}{2}} + \bar{z}^{\frac{3}{2}})/4\alpha^2 - 3(z+\bar{z})/8\alpha^3 - 3(z^{\frac{1}{2}} + \bar{z}^{\frac{1}{2}})/8\alpha^4,$$

on using (16·5) and (16·6) to separate the variables z, $\bar{z}$ in this boundary condition. But if we take $\chi_3 + i\chi_3^* = f(z)$,

$$\frac{\partial\chi_3}{\partial\xi} = \tfrac{1}{2}f'(z)\frac{dz}{d\zeta} + \tfrac{1}{2}\bar{f}'(\bar{z})\frac{d\bar{z}}{d\bar{\zeta}},$$

and if we put
$$\tfrac{1}{2}f'(z)\frac{dz}{d\zeta} = -z^{\frac{3}{2}}/4\alpha^2 - 3z/8\alpha^3 - 3z^{\frac{1}{2}}/8\alpha^4 + iC,$$

where C is a real constant, the boundary condition is satisfied. Using (16·1), we find

$$f(z) = z/4\alpha^2 + 3z^{\frac{1}{2}}/4\alpha^3 + (3\log z)/8\alpha^4,$$

dropping the term in C, which gives rise to a physically inadmissible solution; hence, substituting for α from (16·3), we have

$$\Omega_3 = \chi_3 + i\chi_3^* = az/2 + 3\,.\,2^{-\frac{1}{2}}a^{\frac{3}{2}}z^{\frac{1}{2}} + (3a^2\log z)/2. \quad (16·13)$$

This will be a suitable solution provided the combination $\chi_1 - h\chi_3$ contains no term arising from the logarithms. Now the critical portion of $\chi_1 - h\chi_3$ is

$$\{3a^2h/2 - 5a^3/4\}R\log z,$$

which vanishes if $h = 5a/6$, which is the case, so that our solutions (16·7), (16·9)–(16·13) are satisfactory in every respect.

The torsion and flexure moment integrals

We have the following results for the cardioid cross-section:

$$\left.\begin{aligned} h = 5a/6, \quad S = 3\pi a^2/2, \quad \int (x^2+y^2)\,dS = 35\pi a^4/16, \\ \int y^2\,dS = 21\pi a^4/32, \quad \int xy^2\,dS = 33\pi a^5/64, \quad \int x^3\,dS = 135\pi a^5/64. \end{aligned}\right\} \qquad (16\cdot14)$$

Using (8·3) and (8·4), we find by elementary integration

$$L_2' = 107\pi a^5/64, \quad M_1' = -77\pi a^5/64, \quad M_3' = -9\pi a^4/8,$$

whence, from (7·4) and (16·14), we have

$$L_2 = 37\pi a^5/32, \quad M_1 = 29\pi a^5/32, \quad M_3 = 17\pi a^4/16. \qquad (16\cdot15)$$

The associated flexural torsion and the centre of flexure

These values of M_1, M_3 and L_2 give for the associated twist τ', from (11·13) and (16·14)

$$\tau' = -(W'a/EI')\,(3+4\eta)/51 \qquad (16\cdot16)$$

and from (11·14) $$f_0/a = (111+2\sigma)/126; \qquad (16\cdot17)$$

hence $$(f_0-h)/a = (3+\sigma)/63,$$

and so when the load-point is the centroid it appears that the general twist takes the line of cusps in the direction of the flexing force. Incidentally in this case the anticlastic displacements enhance the visible effect of the twist as indicated by the relative twist of the axis of symmetry, since the so-called "total torsion" is

$$-(W'a/EI')\,(6+25\eta)/102.$$

The relative position of the centre of flexure F and the centroid G is shown in fig. 4*a*; note that the sign of the associated twist τ' is that of the moment of W' localized at G about F.

Comparison with previous solution

The result (16·16) agrees numerically with that due to Shepherd (1936, p. 507) but the situation with regard to the sign needs some consideration. His notation with regard to axes, description of the beam, and loading differs from ours. He takes $z = 0$ as the loaded end-section with the load along the *negative* y-axis, but nowhere states explicitly the location of his cylinder, i.e. whether the elastic material is given by $z>0$, or by $z<0$ (see figs. 4*b*, 4*c*). He finds (for $\eta = 0\cdot3$) a position of the flexural centre on the side of the centroid remote from the cusp, with a positive associated twist, a state of affairs which is only consonant with the load W along the *positive* y-axis as indicated in figs. 4*b*, 4*c*.

In his paper Shepherd corrects the sign of the associated twist in a former paper

(1932, p. 607) in which he discusses the uni-axial cross-section in the form of a circle with a radial slit of any depth, extending inwards from the circumference. In that paper he stated that "the twist is such that the side of the shaft in which the slit is cut is turned in the direction of the bending force", and this result is in agreement with our result for the cardioid and also that for the circle with complete radial slit obtainable from the entry for $\beta=\pi$ in column 7 of Table I. The effect of his correction in his second paper would be to reverse this result so that we are apparently no longer in agreement. When we examine the stresses given in his second paper on pp. 500, 501 we find

$$\int \widehat{yz}\, dS = \int y \frac{\partial \widehat{zz}}{\partial z}\, dS = \frac{W}{I}\int y^2\, dS = W$$

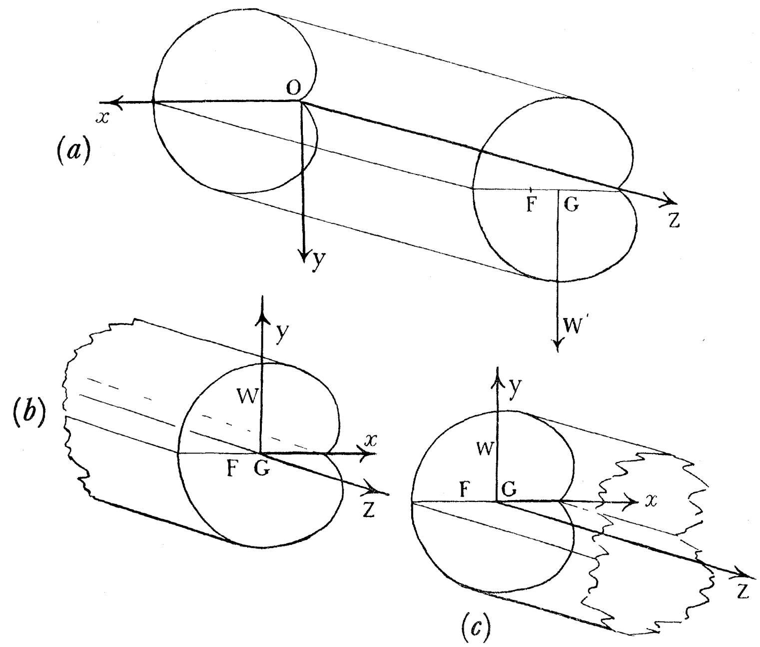

FIG. 4. (*a*) Cardioid cross-section, scheme of axes used in § 16. (*b*) and (*c*) Schemes of axes for comparison with a previous solution.

and the discrepancies disappear if his analysis is regarded as for the beam $z<0$, with the load at the section $z=0$, but along the *positive* y-axis, and not the negative y-axis as stated by him. His correction in the second paper should still stand, of course, and then his results for the cardioid section and the now doubly corrected results for the circular shaft with radial slit are brought into line with Young, Elderton and Pearson's result for the circular shaft with complete radial slit.

17. Cross-section a right-angled isosceles triangle

The flexure problem for a beam whose cross-section is an isosceles right-angled triangle has been considered by Seth (1933), and the corresponding torsion problem by Galerkin (1919) and Kolossoff (1924). The object of this section is to find the associated twist and the centre of flexure for this cross-section, which was not attempted by Seth. We shall need the torsion moment also, which was not evaluated by Kolossoff; the writer has been unable to consult the work of Galerkin.

The canonical flexure functions

Here the equation of the boundary is (see fig. 5)

$$(x-a)(x^2-y^2) = 0,$$

or

$$(z+\bar{z}-2a)(z^2+\bar{z}^2) = 0,$$

whence

$$z\bar{z}^2+\bar{z}z^2 = 2a(z^2+\bar{z}^2)-z^3-\bar{z}^3. \qquad (17\cdot1)$$

The x-axis divides the cross-section into two parts each of which is still a right-angled triangle, the boundaries of the three triangles being included in

$$y(x-a)(x^2-y^2) = 0,$$

or

$$(z-\bar{z})(z+\bar{z}-2a)(z^2+\bar{z}^2) = 0,$$

whence

$$z\bar{z}^2-\bar{z}z^2 = \bar{z}^3-z^3+(z^4-\bar{z}^4)/2a, \qquad (17\cdot2)$$

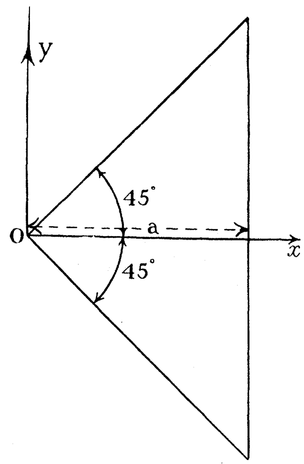

FIG. 5. Cross-section a right-angled isosceles triangle.

so that round the boundary of our cross-section we may replace $\bar{z}z^2$ by a separable form in z and $\bar{z}$ and the boundary condition

$$\psi_1+i\psi_2 = \bar{z}^3/12+\bar{z}z^2/4$$

becomes, on using (17·1) and (17·2),

$$\psi_1+i\psi_2 = a(z^2+\bar{z}^2)/4-\bar{z}^3/6-(z^4-\bar{z}^4)/16a.$$

With the method of § 14, we have at once from (14·13)

$$f(z) = az^2/4-z^4/16a,$$

$$F(z) = az^2/4-z^3/6+z^4/16a,$$

and (14·14) and (14·15) give us

$$\omega_1 = \phi_1+i\psi_1 = i\{-z^3/6+az^2/2\}, \qquad (17\cdot3)$$

$$\omega_2 = \phi_2+i\psi_2 = z^3/6-z^4/8a. \qquad (17\cdot4)$$

For the flexure functions χ_2 and $\chi_4 = \chi_1-h\chi_3$, we proceed in the manner used by Seth in finding the Saint-Venant flexure functions. Since χ_2 satisfies

$$\frac{\partial}{\partial n}(\chi_2-y^3/3) = 0$$

round the boundary, the boundary conditions to be satisfied are

$$\frac{\partial \chi_2}{\partial x} = 0 \quad \text{over } x = a,$$

$$\frac{\partial \chi_2}{\partial x} \pm \left(\frac{\partial \chi_2}{\partial y} - y^2\right) = 0 \quad \text{over } y = \pm x.$$

Assuming a plane harmonic solution

$$\chi_2 = Axy + B(y^3/3 - x^2y),$$

we find the boundary conditions are satisfied by

$$A = a, \quad B = \tfrac{1}{2};$$

hence

$$\chi_2 = axy + (y^3/3 - x^2y)/2. \tag{17·5}$$

Since $h = 2a/3$, χ_4 satisfies

$$\frac{\partial}{\partial n}\{\chi_4 - x^3/3 + a(x^2+y^2)/3\} = 0$$

round the boundary, so that the boundary conditions are

$$\frac{\partial \chi_4}{\partial x} = a^2/3 \quad \text{over } x = a,$$

$$\left\{\frac{\partial \chi_4}{\partial x} - x^2 + 2ax/3\right\} \pm \left\{\frac{\partial \chi_4}{\partial y} - 2ay/3\right\} = 0 \quad \text{along } y = \pm x.$$

Assuming a plane harmonic solution

$$\chi_4 = A(x^2-y^2) + B(x^3 - 3xy^2) + C(x^4 - 6x^2y^2 + y^4),$$

we find the boundary conditions are satisfied by

$$A = -a/3, \quad B = \tfrac{1}{6}, \quad C = -1/24a,$$

so that

$$\chi_4 = \chi_1 - h\chi_3 = -a(x^2-y^2)/3 + (x^3 - 3xy^2)/6 - (x^4 - 6x^2y^2 + y^4)/24a. \tag{17·6}$$

There remains the torsion function, and to find this we write

$$\psi_3 = \psi_3' + ax - (x^2 - y^2)/2,$$

then ψ_3' is a plane harmonic function which has to satisfy the boundary conditions

$$\psi_3' = x^2 - ax + \text{const.} \quad \text{along } y = \pm x$$
$$= \text{const.} \quad \text{along } x = a.$$

Consider the plane harmonic function

$$\psi_n = \cos m(x-a+y)\sinh m(x-a-y) + \cos m(x-a-y)\sinh m(x-a+y), \tag{17·7}$$

which reduces to zero over $x = a$, and if

$$\cos ma = 0, \quad \text{i.e. } m = (n+\tfrac{1}{2})\,\pi/a, \tag{17·8}$$

reduces to $\qquad (-1)^{n+1}\sinh ma \sin 2mx \quad \text{over } y = \pm x.$

Hence we write $$\psi_3' = \sum_{n=0}^{\infty} a_n \psi_n,$$

and determine the coefficients a_n from the boundary condition

$$f(x) = x^2 - ax = \sum_{n=0}^{\infty} a_n (-1)^{n+1} \sinh ma \sin 2mx + \text{const.}$$

From the Fourier expansion as a sine series for the range $0 \leqslant x \leqslant 2a$, of the function $F(x)$ defined by

$$F(x) = f(x), \quad \text{for } 0 \leqslant x \leqslant a,$$
$$= -f(x-2a), \quad \text{for } a \leqslant x \leqslant 2a,$$

we have $$x^2 - ax = -(8a^2/\pi^3) \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin(2n+1)\frac{\pi x}{a},$$

whence $$a_n = (-1)^n / am^3 \sinh ma. \tag{17·9}$$

Hence $$\omega_3 = \phi_3 + i\psi_3 = iaz - iz^2/2 + \sum_{n=0}^{\infty} a_n \omega_n, \tag{17·10}$$

where $$\omega_n = \phi_n + i\psi_n = \sin[m(1+i)\,(z-a)] - \sin[m(1-i)\,(z-a)] \tag{17·11}$$

from (17·7).

This result is equivalent, due regard being paid to the change of axes necessary for comparison, to Seth's transcription of Kolossoff's result.

The torsion and flexure moment integrals

For this cross-section we have the following results:

$$\left.\begin{aligned} h = 2a/3, \quad \int x^2\,dS = a^4/2, \quad \int y^2\,dS = a^4/6, \\ S = a^2, \quad \int xy^2\,dS = 2a^5/15, \quad \int x^3\,dS = 2a^5/5. \end{aligned}\right\} \tag{17·12}$$

From (7·5) and (17·5)

$$L_2' = a\int (x^2 - y^2)\,dS - \tfrac{1}{2}\int (x^3 - 3xy^2)\,dS = a^5/3,$$

on using (17·12), and then from (7·4) and (17·12) we find

$$L_2 = a^5/5. \tag{17·13}$$

Again from (8·4) and (17·3)

$$M_1' = -a\int (x^2 - y^2)\,dS + \tfrac{3}{2}\int (x^3 - 3xy^2)\,dS = -a^5/3,$$

on using (17·12), and then from (7·4) and (17·12) we have

$$M_1 = a^5/15. \tag{17·14}$$

Writing $\omega_3 = \omega_{31} + \omega_{32}$, with $M'_3 = M'_{31} + M'_{32}$ to correspond,

where, from (17·10), $\omega_{31} = iaz - iz^2/2, \quad \omega_{32} = \sum_{n=0}^{\infty} a_n \omega_n,$

we then have, from (8·4),

$$M'_{31} = -a\int x\, dS + \int (x^2 - y^2)\, dS = -a^4/3$$

on using (17·12).

Also from (8·7) $M'_{32} = R\int \tfrac{1}{2} z\bar{z} \frac{\partial \omega_{32}}{\partial s}\, ds = R\sum_{n=0}^{\infty} a_n \int \tfrac{1}{2} z\bar{z} \frac{\partial \omega_n}{\partial s}\, ds,$

where the line integral is taken round the cross-section, whence

$$M'_{32} = \sum_{n=0}^{\infty} m a_n \{4I_1 \cosh ma + 4(-1)^n I_2 - I_3 - I_4\},$$

where I_1, I_2, I_3, I_4 are given by

$$I_1 = \int_0^a x^2 \cos m(2x-a)\, dx = (-1)^n (a/2m - 1/2m^3),$$

$$I_2 = \int_0^a x^2 \sinh m(2x-a)\, dx = (a^2/2m) \cosh ma - (a/2m^2) \sinh ma,$$

$$\begin{aligned} I_3 + iI_4 &= \int_{-a}^{+a} (a^2+y^2) \cos m(1-i)\, y\, dy \\ &= (-1)^n \{(2a^2/m + 1/m^3) \cosh ma - (2a/m^2) \sinh ma\} \\ &\qquad + i(-1)^n (2a^2/m - 1/m^3) \cosh ma, \end{aligned}$$

leading to $M'_{32} = \sum_{n=0}^{\infty} (2a_n/m^2)(-1)^{n+1} \cosh ma,$

where a_n is given by (17·9) or

$$M'_{32} = -2 \sum_{n=0}^{\infty} (\coth ma)/am^5.$$

Hence, finally, from (7·4) and (17·12) we have

$$M_3 = a^4 \left\{ \frac{1}{3} - \frac{2^6}{\pi^5} \sum_{n=0}^{\infty} \frac{\coth(n+\frac{1}{2})\pi}{(2n+1)^5} \right\} = 0·1043586a^4. \tag{17·15}$$

The coefficient of a^4 would appear to agree with a result 4/38·3, quoted by Timoshenko (1934, p. 251), possibly taken from Galerkin's paper, which the writer has not been able to consult.

The associated twist and the centre of flexure

Using (17·13), (17·14) and (17·15) in (11·13), we find with the aid of I' from (17·12) that the associated twist τ' is given by

$$\tau' = (W'a/EI')\{0{\cdot}2129410 + 0{\cdot}0278437\eta\}, \tag{17·16}$$

whilst from (11·14) the abscissa of the centre of flexure is given by

$$f_0/a = 0{\cdot}6 - 0{\cdot}0087171\sigma. \tag{17·17}$$

This completes Seth's solution by relating the constants of the solution completely with the external force system.

It appears that the cross-section twists so that the right-angle edge of the cylinder is bent away from the flexing force, and although the anticlastic displacements tend somewhat to mask the visible effect of the twisting made evident by the relative displacements of the ends of the axis of symmetry, the so-called "total torsion" of § 13, given by equation (12·2), being

$$(W'a/EI')\{0{\cdot}2129410 - 0{\cdot}1388230\eta\}, \quad 0 < \eta < \tfrac{1}{2},$$

it is clear that for no possible value of η can this become negative.

18. Cross-section a loop of the lemniscate of Bernoulli

The object of this section is to give the correct solution for the cross-section which is a loop of the lemniscate of Bernoulli, to replace the incorrect solution given by Young, Elderton and Pearson (1918, p. 61). It illustrates the method of § 14, and is in fact the problem which led to the discovery of that method.

Consider the system of orthogonal curvilinear co-ordinates ξ, η defined by

$$(z+c)(z-c) = c^2e^{2\zeta}, \tag{18·1}$$

where $\zeta = \xi + i\eta$, $z = x + y$. Writing

$$z - c = r_1 e^{i\theta_1}, \quad z + c = r_2 e^{i\theta_2},$$

(18·1) is equivalent to $\qquad r_1 r_2 = c^2 e^{2\xi}, \quad \theta_1 + \theta_2 = 2\eta,$

or, in cartesian co-ordinates

$$(x^2 - y^2 - c^2)^2 + 4x^2y^2 = c^4e^{4\xi},$$

$$x^2 - y^2 - c^2 = 2xy \cot 2\eta,$$

so that $\xi = \text{const.}$ gives a family of quartic curves (Cassini ovals), called by Basset (1884, p. 242) confocal lemniscates, with real foci at $z = \pm c$, and $\eta = \text{const.}$ gives the orthogonal family of curves, which are rectangular hyperbolas, each of which passes through the foci of the lemniscates. At infinity, $\xi = \infty$; at either focus $\xi = -\infty$. From (18·1), the curve $\xi = 0$ is given by

$$z = re^{i\theta} = c(2\cos\eta)^{\frac{1}{2}}e^{\frac{1}{2}i\eta}, \tag{18·2}$$

or

$$r^2 = 2c^2\cos 2\theta, \tag{18·3}$$

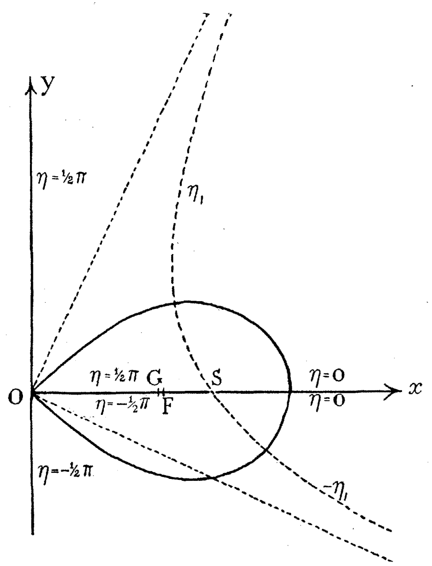

FIG. 6. Cross-section one loop of the lemniscate of Bernoulli. (G = centroid, S = focus, F = centre of flexure.)

in polar co-ordinates, so that the curve $\xi = 0$ is the lemniscate of Bernoulli. This separates the lemniscates for which $\xi > 0$ and which are single ovals from the lemniscates for which $\xi < 0$, each of which consists of a pair of ovals, one about each of the two foci $z = \pm c$. If we take the portions of a hyperbola in the four cartesian quadrants to correspond to values η, $\eta + \pi/2$, $\eta - \pi$, $\eta - \pi/2$ respectively, there is a discontinuity of π in crossing the x-axis between $x = +c$ and $x = -c$, and a discontinuity of 2π along the x-axis for $x < -c$. One loop of the lemniscate of Bernoulli encloses the region

$$-\infty \leqslant \xi \leqslant 0, \quad -\pi/2 \leqslant \eta \leqslant \pi/2,$$

and this is the boundary of the cross-section considered (see fig. 6).

Along the lemniscate $\xi = 0$ we have $z^2 - c^2 = c^2 e^{2i\eta}$, so that the equation of the boundary in conjugate complex variables z and $\bar{z}$ is

$$(z^2 - c^2)(\bar{z}^2 - c^2) = c^4, \tag{18·4}$$

from which we have that, along the boundary,

$$z^2 + \bar{z}^2 = z^2\bar{z}^2/c^2 = z^4/(z^2 - c^2) = \bar{z}^4/(\bar{z}^2 - c^2). \tag{18·5}$$

The canonical flexure functions

From (18·4) we have $\quad (z^2 - c^2)(\bar{z} - c) = c^4/(\bar{z} + c)$

so that round the boundary we can express $\bar{z}z^2$ in separable form, the boundary condition for ψ_1 and ψ_2

$$\psi_1 + i\psi_2 = \bar{z}^3/12 + \bar{z}z^2/4 + \text{const.},$$

becoming $\quad \psi_1 + i\psi_2 = \bar{z}^3/12 + c^2\bar{z}/4 + c(z^2 - c^2)/4 + c^4/4(\bar{z} + c) + \text{const.};$

hence by the method of § 14, comparing with (14·13), we have

$$f(z) = cz^2/4,$$

$$F(\bar{z}) = \bar{z}^3/12 + c^2\bar{z}/4 + c^4/4(\bar{z} + c),$$

so that from equations (14·14) and (14·15) we have at once

$$\omega_1 = \phi_1 + i\psi_1 = i\{z^3/12 + cz^2/4 + c^2z/4 + c^4/4(z+c)\}, \tag{18·6}$$

$$\omega_2 = \phi_2 + i\psi_2 = -z^3/12 + cz^2/4 - c^2z/4 - c^4/4(z+c), \tag{18·7}$$

which give perfectly satisfactory solutions across the cross-section, since the point $z = -c$ falls outside the cross-section.

The canonical flexure functions χ_1 and χ_2 have to satisfy the boundary condition

$$\frac{\partial}{\partial n}\{\chi_1 + i\chi_2 - \bar{z}^3/12 - \bar{z}z^2/4\} = 0,$$

or since $\partial/\partial\xi = \partial/\partial\zeta + \partial/\partial\bar{\zeta}$, the boundary condition can be written

$$\frac{\partial}{\partial \xi}(\chi_1 + i\chi_2) = \tfrac{1}{4}(z^2 + \bar{z}^2)\frac{d\bar{z}}{d\bar{\zeta}} + \tfrac{1}{2}z\bar{z}\frac{dz}{d\zeta},$$

but

$$z\frac{dz}{d\zeta} = z^2 - c^2, \quad \bar{z}\frac{d\bar{z}}{d\bar{\zeta}} = \bar{z}^2 - c^2;$$

hence the boundary condition becomes

$$\frac{\partial}{\partial \xi}(\chi_1 + i\chi_2) = (z^2 + \bar{z}^2)(\bar{z}^2 - c^2)/4\bar{z} + \bar{z}(z^2 - c^2)/2$$

or

$$\frac{\partial}{\partial \xi}(\chi_1 + i\chi_2) = \bar{z}^3/4 + c(z^2 - c^2)/2 + c^4/2(\bar{z} + c)$$

on making use of (18·5). But if $\chi_1 + i\chi_2$ is given by (14·19), then the boundary condition is

$$\frac{\partial}{\partial \xi}(\chi_1 + i\chi_2) = f'(z)\frac{dz}{d\zeta} + F'(\bar{z})\frac{d\bar{z}}{d\bar{\zeta}},$$

and so we may write

$$f'(z)\frac{dz}{d\zeta} = c(z^2 - c^2)/2 + c^3C,$$

$$F'(\bar{z})\frac{d\bar{z}}{d\bar{\zeta}} = \bar{z}^3/4 + c^4/2(\bar{z} + c) - c^3C,$$

where C is an arbitrary constant, and these lead to

$$f(z) = cz^2/2 - \tfrac{1}{2}Cc^3\log(z^2 - c^2),$$

$$F(z) = z^3/12 + c^2z/4 - c^4/4(z+c) + (C/2 - \tfrac{1}{4})\,c^3\log(z+c) + (C/2 + \tfrac{1}{4})\,c^3\log(z-c).$$

Now Ω_1 and Ω_2 are given by (14·20) and (14·21), and χ_2 and $\chi_1 - h\chi_3$ have to be physically admissible solutions, hence we find that in order that χ_2 shall behave satisfactorily at the point $z = c$, we must take $C = -\tfrac{1}{4}$, and we have

$$\Omega_2 = \chi_2 + i\chi_2^* = i\{z^3/12 - cz^2/4 + c^2z/4 - c^4/4(z+c) - \tfrac{1}{2}c^3\log(z+c)\}, \tag{18·8}$$

$$\Omega_1 = \chi_1 + i\chi_1^* = z^3/12 + cz^2/4 + c^2z/4 - c^4/4(z+c) - \tfrac{1}{4}c^3\log(z+c) + \tfrac{1}{4}c^3\log(z-c), \tag{18·9}$$

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of which the last term is unsuitable for our problem and we shall have to show that the logarithmic term disappears from the combination $\chi_1 - h\chi_3$.

Now consider χ_3, this must satisfy the boundary condition

$$\frac{\partial \chi_3}{\partial \xi} = \left(\frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \bar{\zeta}}\right) \tfrac{1}{2} z\bar{z} = \tfrac{1}{2}\bar{z}\frac{dz}{d\zeta} + \tfrac{1}{2}z\frac{d\bar{z}}{d\bar{\zeta}}$$

or

$$\frac{\partial \chi_3}{\partial \xi} = \tfrac{1}{2}c^2 z\bar{z} = c^2 \cos \eta$$

on making use of (18·5) and (18·2).

We now come to a very interesting point; at first sight it would appear that the appropriate solution for χ_3 must be $c^2 e^{\xi} \cos \eta$, since this is satisfactory at all points of the cross-section including the focus $z = c$, as far as the displacements are concerned. But we must remember that such a solution has also to be satisfactory as regards the stresses, i.e. the derivatives of χ_3 have to be satisfactory at all points of the cross-section as well as χ_3 itself. In the present case it is clear that if (18·9) is to stand, then χ_3 must possess a logarithmic term to make the combination $\chi_1 - h\chi_3$ a physically admissible solution, and its derivatives must be finite at all points across the cross-section, otherwise we have non-admissible infinite stresses. Now if

$$d\zeta\, d\bar{\zeta} = J^2\, dz\, d\bar{z} = J^2\, ds^2,$$

we find

$$J^2 = z\bar{z}/c^4 e^{4\xi} \quad \text{or} \quad J = re^{2\xi}/c^2,$$

so that J becomes infinite at the focus $z = c$ inside the boundary $\xi = 0$, since at this point $\xi = -\infty$. But if $(l, m, 0)$ are the direction cosines of the normal to a curve $\xi = \text{const.}$ at the point $P\ (\xi, \eta)$, then

$$\frac{\partial \chi_3}{\partial x} = J\left(l\frac{\partial \chi_3}{\partial \xi} - m\frac{\partial \chi_3}{\partial \eta}\right),$$

$$\frac{\partial \chi_3}{\partial y} = J\left(l\frac{\partial \chi_3}{\partial \eta} + m\frac{\partial \chi_3}{\partial \xi}\right).$$

Hence if the stresses are not to become infinite at the focus, the derivatives of χ_3, expanded in the form $\Sigma A_n e^{n\xi} \cos n\eta$ as a sum of harmonic functions, cannot contain any lower power of e^{ξ} than $e^{2\xi}$. This forces us to reject the solution $c^2 e^{\xi} \cos \eta$, even though it satisfies the boundary condition. If, however, we use the Fourier series for $\cos \eta$

$$\cos \eta = (2/\pi) - (4/\pi) \sum_{n=1}^{\infty} (-1)^n \cos 2n\eta/(4n^2 - 1) \tag{18·10}$$

valid for $-\pi/2 \leqslant \eta \leqslant \pi/2$, then the appropriate solution for χ_3 which satisfies the boundary condition is

$$\chi_3 = (2c^2/\pi)\, \xi - (4c^2/\pi) \sum_{n=1}^{\infty} (-1)^n e^{2n\xi} \cos 2n\eta / 2n(4n^2 - 1), \tag{18·11}$$

of which the only non-admissible term is the first, which becomes infinite at the focus $z = c$, and is the real part of $(2c^2/\pi)\,\zeta = (c^2/\pi)\log\{(z^2-c^2)/c^2\}$. The critical terms in the combination $\chi_1 - h\chi_3$ are accordingly

$$R\{(c^2h/\pi)\log(z^2-c^2)-(c^3/4)\log(z-c)\},$$

which can give rise to no infinity at $z = c$ if $h = \pi c/4$, and this is found to be so; hence our solutions for χ_1 and χ_3 are entirely satisfactory.

We have $$\Omega_3 = \chi_3 + i\chi_3^* = (2c^2/\pi)\,\zeta + c^2 \sum_{n=1}^{\infty} A_n e^{2n\zeta}, \qquad (18\cdot12)$$

where $$A_n = (2/\pi)\,(-1)^{n+1}\{1/(2n-1)+1/(2n+1)-1/n\}, \qquad (18\cdot13)$$

whence $$\Omega_3 = (2c^2/\pi)\,\{\zeta + 2\sinh\zeta\tan^{-1}e^{\zeta} - \log(1+e^{2\zeta})\},$$

dropping an irrelevant constant. The most useful form for our purpose is however (18·12).

Finally we have to consider the torsion function. The boundary condition for ψ_3 is, from (18·2),

$$\psi_3 = z\bar{z}/2 + \text{const.} = c^2\cos\eta + \text{const.} \quad \text{along } \xi = 0,$$

and using (18·10) for $\cos\eta$, this leads to

$$\psi_3 = (4c^2/\pi) \sum_{n=1}^{\infty} (-1)^{n+1} e^{2n\xi}\cos 2n\eta/(4n^2-1),$$

which satisfies the boundary condition and is satisfactory at all points of the cross-section with regard to the displacements and the stresses.

We have $$\omega_3 = \phi_3 + i\psi_3 = ic^2 \sum_{n=1}^{\infty} B_n e^{2n\zeta}, \qquad (18\cdot14)$$

where $$B_n = (2/\pi)\,(-1)^n\{1/(2n+1)-1/(2n-1)\}, \qquad (18\cdot15)$$

or $$\omega_3 = i(2c^2/\pi)\,2\cosh\zeta\tan^{-1}e^{\zeta} \quad \text{(dropping a constant)},$$

but (18·14) is the more useful form. This completes the canonical flexure functions.

Comparison with previous solutions

These results disagree with the solution due to Young, Elderton and Pearson (1918, p. 64) mentioned in the introductory § 1. Investigation shows that their value of the flexure function ϕ in their equation (105) is unsuitable for the physical problem. Their function ϕ (which is equivalent to a combination of Saint-Venant's flexure function and the associated torsion function) has been chosen to satisfy the boundary conditions and make the displacement w finite at the focus. But, as we have seen, this is not sufficient; we have also to ensure that the stresses are not infinite at the focus. Their solution fails because their function ϕ contains a term $\tau c^2 e^{-\alpha/2}\cos(\beta/2)$, leading to stresses of order $e^{\alpha/2}$ at the focus, where in their notation α becomes infinite.

In our notation $\alpha = -2\xi$, and this incorrect term is precisely of the type $e^{\xi}\cos\eta$ which we have rejected in finding χ_3 and ψ_3.

Our result for the torsion function ψ_3 disagrees also with the solution for the corresponding hydrodynamical problem of the motion of an inviscid liquid in a rotating bowl whose cross-section is the lemniscate of Bernoulli, given by Basset (1884, p. 245). He realized the need for care at the focus, but replaced $\cos\eta$ by the expansion

$$\cos\eta = (\pi/4) - \sum_{n=1}^{\infty}(-1)^n \cos(2n+1)\,\eta/(2n+1),$$

valid only for $-\pi/2<\eta<\pi/2$, i.e. excluding the limits $\eta=\pm\pi/2$, instead of using equation (18·10). The result obtained in this way fails to secure continuity of $J\,\partial\phi_3/\partial\xi$ along the x-axis for $0\leqslant x\leqslant c$.

The torsion and flexure moment integrals

For this cross-section we find from (18·2) and (18·3) the results:

$$\left.\begin{aligned} S = c^2, \quad h = \pi c/4, \quad \int y^2\,dS = c^4(\pi/8-\tfrac{1}{3}), \\ \int x^2\,dS = c^4(\pi/8+\tfrac{1}{3}), \quad \int xy^2\,dS = \pi c^5/64, \quad \int x^3\,dS = 15\pi c^5/64, \end{aligned}\right\} \tag{18·16}$$

in calculating which the substitution $\cos\eta = \sin^2\phi$ will be found useful.

From (8·5), (18·14) and (18·15) we find

$$M_3' = c^4\sum_{n=1}^{\infty} B_n\int_{-\pi/2}^{\pi/2}\cos\eta\,\cos 2n\eta\,d\eta = (2c^4/\pi)\sum_{n=1}^{\infty}\{1/(2n+1)^2-1/(2n-1)^2\}$$

or $$M_3' = -2c^4/\pi,$$

so that from (7·4) and (18·16) we then have

$$M_3 = c^4(\pi^2-8)/4\pi = 0{\cdot}1487784c^4. \tag{18·17}$$

Next we put $\quad \Omega_2 = \Omega_{21}+\Omega_{22}, \quad$ with $L_2' = L_{21}'+L_{22}'$ to correspond,

where, from (18·8),

$$\Omega_{21} = i\{z^3/12 - cz^2/4 + c^2z/4\}, \quad \Omega_{22} = -i\{c^4/4(z+c) + (c^3/2)\log(z+c)\}.$$

Then from (8·3) we find

$$L_{21}' = -\tfrac{1}{4}\int(x^3-3xy^2)\,dS + \tfrac{1}{2}c\int(x^2-y^2)\,dS - \frac{c^2}{4}\int x\,dS$$

or $$L_{21}' = -7\pi c^5/64 + c^5/3,$$

using (18·16).

Again, from (8·8) we find

$$L_{22}' = R\int ic^2\cos\eta\{c^4/4(z+c)^2 - c^3/2(z+c)\}\,dz,$$

but since along the boundary $\xi = 0$, we have

$$1/(z+c) = (z-c)\, e^{-2i\eta}/c^2, \quad z = ce^{\frac{1}{2}i\eta}(2\cos\eta)^{\frac{1}{2}}, \quad z\,dz = c^2 i\, d\eta\, e^{2i\eta},$$

we find

$$L'_{22} = -\frac{c^5}{4}\{2^{\frac{3}{2}} I_{\frac{3}{2},\frac{3}{2}} + 2^{\frac{1}{2}} I_{\frac{1}{2},\frac{5}{2}} + 2^{\frac{3}{2}} I_{\frac{1}{2},\frac{1}{2}} - \tfrac{16}{3}\},$$

where†

$$I_{r,s} = \int_0^{\pi/2} \cos^r\eta \cos s\eta\, d\eta = \frac{\pi\Gamma(r+1)}{2^{r+1}\Gamma\frac{(r+s+2)}{2}\,\Gamma\frac{(r-s+2)}{2}}, \tag{18·18}$$

for r real and greater than -1. On using also

$$\Gamma(n)\,\Gamma(1-n) = \pi \operatorname{cosec} n\pi, \tag{18·19}$$

this gives

$$L'_{22} = -c^5(9\pi-32)/24,$$

and so

$$L'_2 = c^5(\tfrac{5}{3} - 31\pi/64).$$

Hence from (7·4) and (18·16)

$$L_2 = c^5(10-3\pi)/6 = 0{\cdot}0958703c^5. \tag{18·20}$$

Similarly we put

$$\omega_1 = \omega_{11} + \omega_{12}, \quad \text{with } M'_1 = M'_{11} + M'_{12} \text{ to correspond,}$$

where from (18·6),

$$\omega_{11} = i\{z^3/12 + cz^2/4 + c^2z/4\}, \quad \omega_{12} = c^4/4(z+c).$$

Then from (8·4) we have

$$M'_{11} = -\tfrac{1}{4}\int(x^3-3xy^2)\, dS - \tfrac{1}{2}c\int(x^2-y^2)\, dS - \frac{c^2}{4}\int x\, dS$$

or

$$M'_{11} = -c^5(7\pi/64 + \tfrac{1}{3}),$$

on making use of (18·16), and from (8·7) we find

$$M'_{12} = R\left\{-\int \frac{ic^6 \cos\eta}{4(z+c)^2}\, dz\right\}$$

or

$$M'_{12} = 2^{\frac{1}{2}}\frac{c^5}{4}\{I_{\frac{1}{2},\frac{5}{2}} + 2I_{\frac{3}{2},\frac{3}{2}}\} - \frac{c^5}{2}\int_0^{\pi/2}(\cos\eta + \cos 3\eta)\, d\eta,$$

and on using (18·18) and (18·19) we have

$$M'_{12} = c^5(3\pi-8)/24;$$

hence

$$M'_1 = c^5(\pi/64 - \tfrac{2}{3}).$$

From (7·4) and (18·16) we have finally

$$M_1 = c^5(3\pi-8)/12 = 0{\cdot}1187315c^5. \tag{18·21}$$

† This result, originally due to Cauchy, is given in Watson (1914, p. 68). See also Whittaker and Watson (1927, p. 263).

The associated twist and the centre of flexure

The length a of the axis of symmetry of the lemniscate loop is $c\surd 2$; hence, on substituting for c in terms of a in (18·17), (18·20), and (18·21), we find from (11·13) and (18·16)

$$\tau' = -(W'a/EI')\{0{\cdot}0124467 + 0{\cdot}0213877\eta\}. \tag{18·22}$$

From (11·14) the abscissa of the centre of flexure is given by

$$f_0/a = 0{\cdot}5709569 + 0{\cdot}0112036\sigma, \tag{18·23}$$

and since $h/a = 0{\cdot}5553604$, the load at G lies between the sharp edge of the cylinder and the flexural centre. The associated torsion accordingly tends to bend the sharp edge towards the direction in which the flexing force acts, a result of opposite sign to that of Young, Elderton and Pearson, even when we take their method of specifying the twist by the so-called "total torsion" into account, for the anticlastic displacements tend to enhance the visible effect of the twist as indicated by the relative displacements of the ends of the axis of symmetry, and the "total torsion" of (12·2) is

$$-(W'a/EI')\{0{\cdot}0124467 + 0{\cdot}0767481\eta\}.$$

Note on the series expansion of the flexure functions

Young, Elderton and Pearson's function ϕ was obtained as a series derived from a Fourier expansion of the boundary condition, and they failed to realize the simple character of the coefficients in their expansion. One set of their coefficients (1918, p. 64) κ_n we can identify with the coefficients of the canonical flexure function χ_2 in series form as

$$\chi_2 = -c^3 \sum_{n=1}^{\infty} \kappa_n e^{2n\xi} \sin 2n\eta,$$

where
$$\kappa_n = -(2/n\pi)\int_0^{\pi/2} (2\cos\theta)^{\frac{1}{2}} \sin^2\frac{\theta}{2} \sin\frac{3\theta}{2} \sin 2n\theta \, d\theta,$$

and it is not difficult to obtain this explicitly, since

$$\kappa_n = -(2^{\frac{1}{2}}/2n\pi)\{I_{\frac{1}{2},\,2n-\frac{3}{2}} - I_{\frac{1}{2},\,2n+\frac{3}{2}} - I_{\frac{3}{2},\,2n-\frac{1}{2}} + I_{\frac{3}{2},\,2n+\frac{3}{2}}\},$$

and so from (18·18) we find

$$\kappa_1 = -\tfrac{3}{32}, \quad \kappa_2 = \tfrac{1}{32},$$

and
$$\kappa_n = (-1)^n 2^{-5}(n-1)(4n^2-5n+6)\{(2n-4)!/(n+1)!\,n!\} \tag{18·24}$$

for $n \geqslant 3$.

Young, Elderton and Pearson did not effect this integration but computed the coefficients for $n = 1$ to $n = 7$ by numerical integration "after considerable labour". We find for comparison the results in Table II below.

TABLE II

n	κ_n from (18·24)	κ_n computed
1	−0·09375	−0·0937535
2	+0·03125	+0·0312530
3	−0·0058594	−0·0058622
4	+0·0024414	+0·0024437
5	−0·0013184	−0·0013204
6	+0·0008138	+0·0008155
7	−0·0005460	−0·0005482

Evidently the formula by means of which the integral was computed, due to Sheppard (quoted by Pearson 1902, p. 276, case (i) (c)), the subject of integration being tabled to every 5°, is here only reliable to four or five places of the seven figures tabulated.

19. Asymmetric cross-section; the half-loop of Bernoulli's lemniscate

In this section we propose to illustrate the systematic treatment developed in this paper by finding the solution for a completely asymmetric cross-section, namely, one of the two halves of the loop of Bernoulli's lemniscate into which it is divided by the axis of symmetry. This would appear to be the first complete solution for an entirely asymmetric cross-section.*

With the axis of x as the axis of symmetry of a cross-section of uni-axial symmetry, the three functions ϕ_2, χ_1 and χ_3 give the complete solution when the load is along the axis of symmetry. The distribution of stresses will accordingly be symmetrical about the axis of x, and $\widehat{yz}_1$ will be odd in y, with $\widehat{yz}_1 = 0$ along the axis of symmetry $y = 0$. Now consider an elastic cylinder which has for its cross-section one of the two halves into which the axis of symmetry divides the original cross-section of uni-axial symmetry. The boundary condition $l\widehat{xz}+m\widehat{yz} = 0$, which gave rise to the boundary conditions for the canonical flexure functions, reduces to $\widehat{yz} = 0$ along the boundary $y = 0$. Hence $\widehat{yz}_1 = 0$ is the boundary condition for the half-section along the straight boundary $y = 0$ corresponding to the stresses arising from the canonical flexure functions ϕ_2, χ_1 and χ_3. It appears therefore that the functions ψ_2, χ_1, χ_3 which satisfy the boundary conditions round the boundary of the original uni-axial cross-section also satisfy the corresponding boundary conditions all round the boundary of the half-section.

* Since this was written, Seth (1936a) has published some results "On Flexure of Beams of Triangular Cross-section", which include asymmetric cross-sections; these are, however, for the particular case $\eta = \frac{1}{2}$, and in no case does he complete the solution by evaluating the associated twist, which is a very necessary constant of the solution.

The boundary of the half-section obtained in this way from the loop of the lemniscate of Bernoulli consists, from the analytical point of view, of the three portions

$\xi = 0, \quad 0 < \eta < \pi/2,$

$\eta = 0, \quad -\infty < \xi < 0$ (the x-axis from the focus $z = c$ to the leading edge),

$\eta = \pi/2, \quad -\infty < \xi < 0$ (the x-axis from the focus $z = c$ to the trailing edge),

(see fig. 7).

The canonical flexure functions

From the previous remarks we know that ω_2, Ω_1 and Ω_3 are given by (18·7), (18·9) and (18·12), and it is readily verified that these do in fact satisfy the conditions along $y = 0$. We have now to find the three remaining canonical flexure functions ω_1, ω_2 and Ω_2.

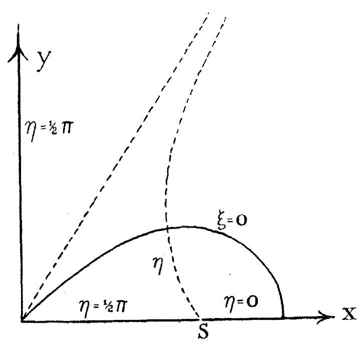

FIG. 7. Asymmetric cross-section bounded by the positive x-axis and the portion of the lemniscate of Bernoulli in the quadrant for which x and y are positive.

If we put $\omega_1 = \omega_{11} + \omega_{12},$

where $$\omega_{11} = iz^3/3, \tag{19·1}$$

then ψ_{12} is a plane harmonic function satisfying

$$\psi_{12} = xy^2 + \text{const.}$$

along the boundary, which becomes

$$\psi_{12} = \text{const.}$$

along $y = 0$, i.e. along $\eta = 0$, and $\eta = \pi/2$, and

$$\psi_{12} = c^3 f(\eta) + \text{const.} \quad \text{along } \xi = 0,$$

where $$f(\eta) = (2\cos\eta)^{\frac{3}{2}} \cos\frac{\eta}{2} \sin^2\frac{\eta}{2}$$

and vanishes for $\eta = 0$ and $\eta = \pi/2$. Hence the boundary condition can be expanded in a Fourier series odd in η, leading to

$$\omega_{12} = c^3 \sum_{n=1}^{\infty} A_n e^{2n\zeta}, \tag{19·2}$$

where $$A_n = (4/\pi) \int_0^{\pi/2} (2\cos\theta)^{\frac{3}{2}} \cos\frac{\theta}{2} \sin^2\frac{\theta}{2} \sin 2n\theta \, d\theta. \tag{19·3}$$

The series (19·2) can be summed, but we find it much more convenient to retain the series expression.

To find the torsion function we write

$$\omega_3 = \omega_{31} + \omega_{32}, \tag{19·4}$$

where $\omega_{31} = iz^2/2$. Then ψ_{32} is a plane harmonic function satisfying the boundary condition

$$\psi_{32} = y^2 + \text{const.}$$

which becomes $\psi_{32} = \text{const.} \quad \text{along } y = 0,$

and $\psi_{32} = c^2 f(\eta) + \text{const.} \quad \text{along } \xi = 0,$

where $f(\eta) = 2\cos\eta \sin^2(\eta/2),$

which vanishes for $\eta = 0$ and $\eta = \pi/2$.

Again, the boundary condition can be expanded in a Fourier series odd in η, leading to

$$\omega_{32} = c^2 \sum_{n=1}^{\infty} C_n e^{2n\zeta}, \tag{19·5}$$

where $\pi C_n = 2\int\{2\cos\theta - 1 - \cos 2\theta\} \sin 2n\theta \, d\theta$

or $\pi C_n = 2/n(4n^2-1) - (1+\cos n\pi)/n(n^2-1).$ (19·6)

Again the series (19·5) can be summed, but the series expression proves to be much more useful.

Finally, χ_2 is a plane harmonic function satisfying the boundary condition

$$\frac{\partial}{\partial n}\{\chi_2 - y^3/3\} = 0,$$

which becomes $\dfrac{\partial \chi_2}{\partial \eta} = 0 \quad \text{along } y = 0,$

and $\dfrac{\partial \chi_2}{\partial \xi} = c^2 f(\eta) \quad \text{along } \xi = 0,$

where $f(\eta) = (2\cos\eta)^{\frac{1}{2}} \sin^2\dfrac{\eta}{2} \sin\dfrac{3\eta}{2}.$

Accordingly we write $\Omega_2 = \chi_2 + i\chi_2^* = c^3\left\{B_0\zeta + \sum_{n=1}^{\infty} B_n e^{2n\zeta}\right\},$ (19·7)

and the boundary conditions will be satisfied if

$$B_n = (2/n\pi)\int_0^{\pi/2} (2\cos\theta)^{\frac{1}{2}} \sin^2\frac{\theta}{2} \sin\frac{3\theta}{2} \cos 2n\theta \, d\theta, \quad n = 1, 2, 3, \ldots \tag{19·8}$$

and $B_0 = (2/\pi)\displaystyle\int_0^{\pi/2} (2\cos\theta)^{\frac{1}{2}} \sin^2\frac{\theta}{2} \sin\frac{3\theta}{2} \, d\theta.$

Using the substitution $\cos\theta = \tan^2\phi$, B_0 is readily found to be

$$B_0 = (2/\pi)\{\tfrac{1}{2}\log(1+\sqrt{2}) - \sqrt{2}/6\}. \tag{19·9}$$

It is clear that this result contains one term, namely $c^3B_0\zeta$, which makes χ_2 become infinite at the focus $z=c$. But χ_2 always occurs in the combination $\chi_2-k\chi_3$, and the critical term in this combination we find from (18·12) to be $c^2\{cB_0-k(2/\pi)\}\xi$, which vanishes if

$$k=c\{\tfrac{1}{2}\log(1+\surd 2)-\surd 2/6\}. \tag{19·10}$$

This is indeed found to be the case; hence the solution (19·7) satisfies all requirements.

The torsion and flexure moment integrals

We have the additional results for this cross-section:

$$\left.\begin{aligned}&S=c^2/2,\quad h=\pi c/4,\quad \int x^2\,dS=c^4(\pi/16+\tfrac{1}{6}),\quad \int y^2\,dS=c^4(\pi/16-\tfrac{1}{6}),\\ &\int xy\,dS=c^4/12,\quad \int xy^2\,dS=\pi c^5/128,\quad \int x^3\,dS=15\pi c^5/128,\\ &\int x^2y\,dS=c^5\{61\surd 2/960-\tfrac{1}{64}\log(1+\surd 2)\},\\ &\int y^3\,dS=c^5\{49\surd 2/320-\tfrac{15}{64}\log(1+\surd 2)\}.\end{aligned}\right\} \tag{19·11}$$

In finding these results the substitutions $\cos\theta=\sin^2\phi$ or $\tan^2\phi$ will be found useful; some of the results are immediately deducible from the corresponding results of § 18.

For an asymmetric cross-section we have to calculate six moment integrals. Consider first the torsion moment for the cross-section. Writing $M'_3=M'_{31}+M'_{32}$, we have from (19·4), using (8·4) and (19·11),

$$M'_{31}=\int(x^2-y^2)\,dS=-c^4/3,$$

and from (8·5)

$$M'_{32}=c^2\int_0^{\pi/2}\cos\eta\left(\frac{\partial\phi_{32}}{\partial\eta}\right)_{\xi=0}d\eta+\tfrac{1}{2}c^2\int_{-\infty}^{0}(1+e^{2\xi})\left(\frac{\partial\phi_{32}}{\partial\xi}\right)_{\eta=0}d\xi+\tfrac{1}{2}c^2\int_0^{-\infty}(1-e^{2\xi})\left(\frac{\partial\phi_{32}}{\partial\xi}\right)_{\eta=\pi/2}d\xi,$$

whence
$$M'_{32}=-\frac{c^4}{2}\sum_{n=1}^{\infty}C_n\left\{\frac{1}{2n-1}-\frac{1}{2n+1}+\frac{1+\cos n\pi}{n+1}\right\}.$$

Hence substituting from (19·6) for C_n, splitting the term of the summation into partial fractions, we have after some reduction using the digamma and trigamma functions, given by (15·25), (15·26) and (15·29),

$$M'_{32}=(c^4/\pi)\{\tfrac{5}{3}\psi(\tfrac{5}{4})+\psi(1)+\psi(\tfrac{3}{4})-\tfrac{17}{12}\psi(\tfrac{1}{2})-\tfrac{9}{4}\psi(\tfrac{3}{2})+\tfrac{1}{4}\psi'(\tfrac{1}{2})-2\},$$

or using (15·31), (15·32) and (15·33)

$$M'_{32}=(c^4/\pi)\{\pi^2/8-\pi/3+\tfrac{1}{6}-\tfrac{2}{3}\log 2\},$$

whence
$$M'_3=(c^4/\pi)\{\pi^2/8-2\pi/3+\tfrac{1}{6}-\tfrac{2}{3}\log 2\}.$$

Hence from (7·4) and (19·11)

$$M_3 = (c^4/\pi)\{\pi^2/4 - 2\pi/3 + \tfrac{1}{6} - \tfrac{2}{3}\log_e 2\} = 0\cdot0246927c^4. \tag{19·12}$$

We next calculate L_1', writing from (18·9),

$$\Omega_1 = \Omega_{11} + \Omega_{12} + \Omega_{13}, \quad \text{with } L_1' = L_{11}' + L_{12}' + L_{13}' \text{ to correspond,}$$

where

$$\Omega_{11} = z^3/12 + cz^2/4 + c^2z/4,$$

$$\Omega_{12} = -c^4/4(z+c) - (c^3/4)\log(z+c),$$

$$\Omega_{13} = (c^3/4)\log(z-c).$$

We may omit Ω_{13} and L_{13}', since χ_{13} disappears from the combination $\chi_1 - h\chi_3$, provided we omit also the annulling term in χ_3 when calculating L_3. Using (8·3) we obtain

$$L_{11}' = \tfrac{1}{4}\int (y^3 - 3x^2y)\,dS - c\int xy\,dS - \frac{c^2}{4}\int y\,dS$$

or

$$L_{11}' = c^5\{11\surd2/960 - \tfrac{1}{12} - \tfrac{7}{64}\log(1+\surd2)\},$$

on making use of (19·11). From (8·8) we find

$$L_{12}' = \int_0^{\surd 2c} \frac{x^2}{8}\left\{\frac{c^4}{(x+c)^2} - \frac{c^3}{x+c}\right\}dx + R\int_0^{\pi/2} \frac{c^2}{4}\cos\eta\left[\frac{dz}{d\eta}\left\{\frac{c^4}{(z+c)^2} - \frac{c^3}{z+c}\right\}\right]_{\xi=0} d\eta.$$

The first integral gives us $\dfrac{c^5}{8}\{1+\surd2 - 3\log(1+\surd2)\}$,

and since along $\xi = 0$ we have

$$1/(z+c) = (z-c)\,e^{-2i\eta}/c^2, \quad z = c^{\frac{1}{2}i\eta}(2\cos\eta)^{\frac{1}{2}}, \quad z\,dz = ic^2e^{2i\eta}\,d\eta,$$

the second integral reduces to

$$\frac{2^{\frac{1}{2}}c^5}{8}\{J_{\frac{1}{2},\frac{1}{2}} + J_{\frac{1}{2},\frac{5}{2}} + 2J_{\frac{3}{2},\frac{3}{2}} - 2^{\frac{3}{2}}J_{1,2}\},$$

where

$$J_{r,s} = \int_0^{\pi/2} (\cos\eta)^r \sin s\eta\,d\eta, \tag{19·13}$$

and we have the particular values

$$J_{1,2} = \tfrac{2}{3}, \quad J_{\frac{1}{2},\frac{5}{2}} = \tfrac{2}{3}, \tag{19·14}$$

$$J_{\frac{1}{2},\frac{1}{2}} = 2^{-\frac{1}{2}}\{\surd2 - \log(1+\surd2)\}, \tag{19·15}$$

$$J_{\frac{3}{2},\frac{3}{2}} = 2^{-\frac{3}{2}}\{5\surd2/3 - \log(1+\surd2)\}, \tag{19·16}$$

using the substitution $\cos\eta = x$ for $J_{1,2}$, and the substitution $\cos\eta = \tan^2\phi$ for the other integrals. Hence the value of the second integral is

$$c^5\{5\surd2/12 - \tfrac{1}{3} - \tfrac{1}{4}\log(1+\surd2)\}.$$

Hence

$$L_{12}' = c^5\{13\surd2/24 - \tfrac{5}{24} - \tfrac{5}{8}\log(1+\surd2)\},$$

and then

$$L_1' = c^5\{531\surd2/960 - \tfrac{7}{24} - \tfrac{47}{64}\log(1+\surd2)\}.$$

Finally from (7·4) and (19·11) we find

$$L_1 = c^5\{37\surd 2/60 - \tfrac{7}{24} - \tfrac{3}{4}\log(1+\surd 2)\} = -0{\cdot}0805984c^5. \tag{19·17}$$

In similar fashion, to calculate M_2, we write, from (18·7),

$$\omega_2 = \omega_{21} + \omega_{22}, \quad \text{with } M_2' = M_{21}' + M_{22}' \text{ to correspond}, \tag{19·18}$$

where $$\omega_{21} = -z^3/12 + cz^2/4 - c^2z/4, \quad \omega_{22} = -c^4/4(z+c). \tag{19·19}$$

Using (8·4) we obtain

$$M_{21}' = -\tfrac{1}{4}\int (y^3 - 3x^2y)\, dS - c\int xy\, dS + \frac{c^2}{4}\int y\, dS,$$

or $$M_{21}' = -c^5\{\tfrac{1}{12} + 11\surd 2/960 - \tfrac{7}{64}\log(1+\surd 2)\},$$

on making use of (19·11). From (8·7) we find

$$M_{22}' = \frac{c^4}{4}\int_0^{\sqrt{2}c} \frac{x^2}{(x+c)^2}\, dx + R\int_0^{\pi/2} \frac{c^6}{4}\cos\eta \left\{\frac{dz}{d\eta}\Big/(z+c)^2\right\}_{\xi=0} d\eta.$$

The first integral gives $$\frac{c^5}{4}\{1 - \log(1+\surd 2)\},$$

and the second integral reduces to

$$c^5\{2^{-\frac{5}{2}} J_{\frac{1}{2},\frac{5}{2}} + 2^{-\frac{3}{2}} J_{\frac{3}{2},\frac{3}{2}} - J_{2,1}\},$$

and since it is readily shown that $J_{2,1} = \frac{1}{3}$, we find from (19·14) and (19·16) that the value of the second integral is

$$c^5\{7\surd 2/24 - \tfrac{1}{3} - \tfrac{1}{8}\log(1+\surd 2)\},$$

so that $$M_{22}' = c^5\{7\surd 2/24 - \tfrac{1}{12} - \tfrac{3}{8}\log(1+\surd 2)\}.$$

Hence $$M_2' = c^5\{269\surd 2/960 - \tfrac{1}{6} - \tfrac{17}{64}\log(1+\surd 2)\},$$

and so finally from (7·4) and (19·11), we have

$$M_2 = c^5\{13\surd 2/30 - \tfrac{1}{6} - \tfrac{1}{2}\log(1+\surd 2)\} = 0{\cdot}0054724c^5. \tag{19·20}$$

Next consider L_3; and write

$$\Omega_3 = \Omega_{31} + \Omega_{32} + \Omega_{33}, \quad \text{with } L_3' = L_{31}' + L_{32}' + L_{33}' \text{ to correspond},$$

where from (18·12),

$$\Omega_{31} = c^2 \sum_{n=1}^{\infty} A_n e^{2n\zeta}, \quad \Omega_{32} = (c^2/\pi)\log(z+c), \quad \Omega_{33} = (c^2/\pi)\log(z-c).$$

We can omit Ω_{33} and L_{33} as explained when considering L_1. Using (8·6) we find

$$L_{31}' = \int_0^{\pi/2} c^2\cos\eta \left(\frac{\partial\chi_{31}}{\partial\eta}\right)_{\xi=0} d\eta + \tfrac{1}{2}c^2\int_{-\infty}^{0} (1+e^{2\xi})\left(\frac{\partial\chi_{31}}{\partial\xi}\right)_{\eta=0} d\xi + \tfrac{1}{2}c^2\int_0^{-\infty} (1-e^{2\xi})\left(\frac{\partial\chi_{31}}{\partial\xi}\right)_{\eta=\pi/2} d\xi$$

or $$L_{31}' = -\frac{c^4}{2}\sum_{n=1}^{\infty} A_n \left\{\frac{1}{2n-1} - \frac{1}{2n+1} + \frac{1+\cos n\pi}{n+1}\right\}.$$

Substituting for A_n from (18·13) and resolving into partial fractions, we express the result in terms of the digamma function and the allied function $\beta(z)$ defined by (Adams 1922, p. 133)

$$\beta(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+z} = \tfrac{1}{2}\left\{\psi\left(\frac{1+z}{2}\right) - \psi\left(\frac{z}{2}\right)\right\}, \tag{19·21}$$

which satisfies the relations

$$\beta(1+z) = 1/z - \beta(z), \quad \beta(1-z) = \pi \operatorname{cosec} \pi z - \beta(z). \tag{19·22}$$

Hence we obtain

$$L'_{31} = (2c^4/\pi)\{\tfrac{1}{2} + \tfrac{1}{4}\beta'(\tfrac{1}{2}) + \tfrac{1}{6}\beta(2) - \tfrac{1}{2}\beta(1) + \tfrac{1}{3}\beta(\tfrac{1}{2}) - \tfrac{1}{2}\psi(\tfrac{3}{2}) - \tfrac{1}{6}\psi(\tfrac{1}{2})\},$$

or using the particular values

$$\beta(1) = \log 2, \quad \beta(\tfrac{1}{2}) = \pi/2, \tag{19·23}$$

and (15·31)
$$L'_{31} = (2c^4/\pi)\left\{\frac{\pi}{6} + \tfrac{2}{3}\log 2 + \tfrac{1}{4}\beta'(\tfrac{1}{2}) - \tfrac{1}{6}\right\}.$$

From the definition of $\beta(z)$, the value of $\beta'(z)$ may be calculated from the tabulated trigamma functions, but if here we write

$$G = -\tfrac{1}{4}\beta'(\tfrac{1}{2}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2},$$

this has been calculated by Glaisher (1877, p. 203) to twenty places of decimals as

$$G = 0{\cdot}91596\ 55941\ 77219\ 01505,$$

which is more than sufficient for our purpose.

From (8·8) we have

$$L'_{32} = (c^2/\pi)\left\{\int_0^{\sqrt{2}c} \frac{x^2}{2(x+c)}\,dx + R\int_0^{\pi/2} c^2 \cos\eta \left[\frac{dz}{d\eta}\Big/(z+c)\right]_{\xi=0} d\eta\right\}$$
$$= (c^4/2\pi)\{1-\sqrt{2}+\log(1+\sqrt{2})\} - (c^4/2\pi)\,2^{\frac{1}{2}}J_{\frac{1}{2},\frac{1}{2}},$$

and so, using (19·15), $\quad L'_{32} = (c^4/\pi)\{\tfrac{1}{2}-\sqrt{2}+\log(1+\sqrt{2})\}.$

Hence, collecting our results, we have from (7·4)

$$L_3 = (c^4/\pi)\{\tfrac{1}{6} - 2G - \sqrt{2} + \pi/3 + \tfrac{4}{3}\log 2 + \log(1+\sqrt{2})\}$$

or
$$L_3 = -0{\cdot}0721643c^4. \tag{19·24}$$

Next we consider M_1; from (19·1) and (19·2) we have

$$M'_1 = M'_{11} + M'_{12},$$

where from (8·4) and (19·11) we have

$$M'_{11} = -\int (x^3 - 3xy^2)\,dS = -3\pi c^5/32,$$

and from (8·7) we find, proceeding as for M'_{32} and L'_{31},

$$M'_{12} = -\tfrac{1}{2}c^5 \sum_{n=1}^{\infty} A_n \left\{\frac{1}{2n-1} - \frac{1}{2n+1} + \frac{1+\cos n\pi}{n+1}\right\}.$$

Using the integral form for A_n of (19·3), and summing the series under the sign of the integration, we have

$$M'_{12} = (c^5/\pi)\int_0^{\pi/2} (2\cos\theta)^{\frac{3}{2}} \cos\frac{\theta}{2} \sin^2\frac{\theta}{2} f(\theta)\, d\theta,$$

where

$$\begin{aligned} f(\theta) &= R\sum_{n=1}^{\infty} 2ie^{2in\theta}\left\{\frac{1}{2n-1} - \frac{1}{2n+1} + \frac{1+\cos n\pi}{n+1}\right\} \\ &= R-\left\{2i + 2\sin\theta \log\frac{1+e^{i\theta}}{1-e^{i\theta}} - 2ie^{-2i\theta}\log\frac{1+e^{2i\theta}}{1-e^{2i\theta}}\right\} \\ &= 2\sin\theta\log\tan\frac{\theta}{2} - 2\sin 2\theta \log\tan\theta - \pi\cos 2\theta, \end{aligned}$$

so that we can separate M'_{12} into the two integrals M'_{121}, M'_{122}, where

$$\begin{aligned} M'_{121} &= -c^5\int_0^{\pi/2} (2\cos\theta)^{\frac{3}{2}} \cos\frac{\theta}{2}\sin^2\frac{\theta}{2}\cos 2\theta\, d\theta \\ &= 2^{-\frac{3}{2}}c^5\{I_{\frac{3}{2},\frac{1}{2}} + I_{\frac{3}{2},\frac{7}{2}} - I_{\frac{3}{2},\frac{3}{2}} - I_{\frac{3}{2},\frac{5}{2}}\} = \pi c^5/128, \end{aligned}$$

using the notation and results of (18·18), and

$$M'_{122} = (2c^5/\pi)\int_0^{\pi/2} (2\cos\theta)^{\frac{3}{2}}\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}\left\{\sin\theta\log\tan\frac{\theta}{2} - \sin 2\theta\log\tan\theta\right\} d\theta.$$

To evaluate this we first put $\cos\theta = \tan^2\phi$, whence

$$2\log\tan(\theta/2) = \log\cos 2\phi, \quad 2\log\tan\theta = \log\cos 2\phi - 4\log\sin\theta,$$

and putting $\tan\phi = t$, $\sec\phi = s$ for brevity, we have

$$M'_{122} = (2c^5/\pi)\Big[I_1 - I_2\Big]_0^{\pi/4},$$

where

$$I_1 = \int 8st^6(1-t^4)\log\sin\phi\, d\phi,$$

$$I_2 = \int st^4(1-t^4)\,(2t^2-1)\log\cos 2\phi\, d\phi.$$

Now since

$$n\int st^n\, d\phi = st^{n-1} - (n-1)\int st^{n-2}\, d\phi,$$

we deduce, for n even,

$$\int st^n\, d\phi = \frac{1}{n}st^{n-1} - \frac{(n-1)}{n(n-2)}st^{n-3} + \frac{(n-1)(n-3)}{n(n-2)(n-4)}st^{n-5} - \ldots + (-1)^{\frac{n}{2}+1}\frac{1.3.5.\ldots.(n-1)}{2.4.6.\ldots.n}\{st - \log(s+t)\}. \quad (19·25)$$

Integrating I_2 by parts, taking $\log\cos 2\phi$ as one member, using (19·25) with different values of n as required, we have with a little further reduction of the integral portion remaining

$$I_2 = \{\tfrac{15}{64}st - \tfrac{5}{32}st^3 - \tfrac{3}{40}st^5 + \tfrac{7}{20}st^7 - \tfrac{1}{5}st^9\}\log\cos 2\phi + \tfrac{15}{64}\int \sec\phi \log\sec 2\phi\, d\phi$$
$$+\int\left\{\tfrac{4}{5}st^8 - \tfrac{3}{5}st^6 - \tfrac{3}{10}st^4 + \tfrac{13}{40}st^2 - \tfrac{49}{80}s + \tfrac{49}{80}\frac{s}{1-t^2}\right\}d\phi.$$

Now, since $$\int \frac{s}{1-t^2}\,d\phi = 2^{-\frac{3}{2}}\{\log(1+2^{\frac{1}{2}}\sin\phi) - \log(1-2^{\frac{1}{2}}\sin\phi)\}$$

and $$\log\cos 2\phi = \log(1+2^{\frac{1}{2}}\sin\phi) + \log(1-2^{\frac{1}{2}}\sin\phi),$$

then, using (19·25) we find

$$I_2 = \{\tfrac{15}{64}st - \tfrac{5}{32}st^3 - \tfrac{3}{40}st^5 + \tfrac{7}{20}st^7 - \tfrac{1}{5}st^9 + \tfrac{49}{320}\sqrt{2}\}\log(1+\sqrt{2}\sin\phi) + \tfrac{15}{64}\int\sec\phi\log\sec 2\phi\,d\phi$$
$$+\{\tfrac{15}{64}st - \tfrac{5}{32}st^3 - \tfrac{3}{40}st^5 + \tfrac{7}{20}st^7 - \tfrac{1}{5}st^9 - \tfrac{49}{320}\sqrt{2}\}\log(1-\sqrt{2}\sin\phi)$$
$$-\tfrac{77}{160}\log(s+t) - \tfrac{21}{160}st + \tfrac{47}{240}st^3 - \tfrac{13}{60}st^5 + \tfrac{1}{10}st^7,$$

whence $$\Big[I_2\Big]_0^{\pi/4} = \tfrac{15}{64}K - \tfrac{77}{160}\log(1+\sqrt{2}) + \tfrac{49}{160}\sqrt{2}\log 2 - \tfrac{5}{96}\sqrt{2},$$

where $$K = \int_0^{\pi/4}\sec\phi\log\sec 2\phi\,d\phi.$$

Proceeding in the same elementary fashion with I_1, we have

$$I_1 = \{-\tfrac{4}{5}st^9 + \tfrac{9}{10}st^7 + \tfrac{17}{60}st^5 - \tfrac{17}{48}st^3 + \tfrac{17}{32}st - \tfrac{17}{32}\log(s+t)\}\log\sin\phi$$
$$+\tfrac{1}{10}st^7 - \tfrac{4}{15}st^5 + \tfrac{21}{80}st^3 - \tfrac{13}{60}st - \tfrac{151}{480}\log(s+t) + \tfrac{17}{32}\int t^{-1}\log(s+t)\,d\phi,$$

whence

$$\Big[I_1\Big]_0^{\pi/4} = \tfrac{17}{32}K' + \tfrac{17}{64}\log 2\log(1+\sqrt{2}) - \tfrac{151}{480}\log(1+\sqrt{2}) - \tfrac{269}{960}\sqrt{2}\log 2 - \tfrac{29}{240}\sqrt{2},$$

where $$K' = \int_0^{\pi/4}\cot\phi\log(\sec\phi + \tan\phi)\,d\phi.$$

Hence we have finally

$$M'_{122} = (c^5/\pi)\{\tfrac{17}{16}K' - \tfrac{15}{32}K + \tfrac{17}{32}\log 2\log(1+\sqrt{2}) + \tfrac{1}{3}\log(1+\sqrt{2}) - \tfrac{563}{480}\sqrt{2}\log 2 + \tfrac{11}{80}\sqrt{2}\}.$$

Collecting up the results to give M'_1, and then using (7·4) and (19·11) to give M_1, we find

$$M_1 = (c^5/\pi)\left\{\tfrac{17}{16}K' - \tfrac{15}{32}K + \tfrac{17}{32}\log 2\log(1+\sqrt{2}) + \tfrac{1}{3}\log(1+\sqrt{2})\right.$$
$$\left. - \tfrac{563}{480}\sqrt{2}\log 2 - \tfrac{11}{80}\sqrt{2} + \frac{\pi^2}{32}\right\}. \quad (19{\cdot}26)$$

To find the numerical value of M_1, we have to compute the values of K and K'. This will be carried out after we have calculated the corresponding formula for L_2, the remaining moment integral, which will be found to depend on the values of K and K' also.

From (19·7), we write

$$\Omega_2 = \Omega_{21}+\Omega_{22}+\Omega_{23}, \quad \text{with } L'_2 = L'_{21}+L'_{22}+L'_{23} \text{ to correspond,}$$

where
$$\Omega_{21} = \sum_{n=1}^{\infty} B_n e^{2n\zeta},$$

and
$$\Omega_{22} = (c^3B_0/2)\log(z+c) = (cB_0\pi/2)\,\Omega_{32},$$
$$\Omega_{23} = (c^3B_0/2)\log(z-c).$$

We omit Ω_{23} and L'_{23}, since we omit a term in χ_3 in obtaining L_3 which annuls L'_{23} in the combination L_2-kL_3. We have

$$L'_{22} = (cB_0\pi/2)\,L'_{32} = \frac{c^5}{2}B_0\{\tfrac{1}{2}-\sqrt{2}+\log(1+\sqrt{2})\}$$

or
$$L'_{22} = (c^5/\pi)\,\{\tfrac{1}{3}-\tfrac{1}{12}\sqrt{2}+(\tfrac{1}{4}-\tfrac{2}{3}\sqrt{2})\log(1+\sqrt{2})+\tfrac{1}{2}[\log(1+\sqrt{2})]^2\},$$

using (19·9).

Again using (8·8) we have

$$L'_{21} = c^2\int_0^{\pi/2}\cos\eta\left(\frac{\partial\chi_{21}}{\partial\eta}\right)_{\xi=0}d\eta+\tfrac{1}{2}c^2\int_{-\infty}^{0}(1+e^{2\xi})\left(\frac{\partial\chi_{21}}{\partial\xi}\right)_{\eta=0}d\xi+\tfrac{1}{2}c^2\int_0^{-\infty}(1-e^{2\xi})\left(\frac{\partial\chi_{21}}{\partial\xi}\right)_{\eta=\pi/2}d\xi,$$

or
$$L'_{21} = -\frac{c^5}{2}\sum_{n=1}^{\infty}nB_n\left\{\frac{2}{2n+1}+\frac{2}{2n-1}-\frac{1}{n}-\frac{1}{n+1}+(1)^n\left[\frac{1}{n}-\frac{1}{n+1}\right]\right\}$$
$$= (c^5/\pi)\int_0^{\pi/2}(2\cos\theta)^{\frac{1}{2}}\sin^2\frac{\theta}{2}\sin\frac{3\theta}{2}f(\theta)\,d\theta,$$

using (19·8), and writing

$$f(\theta) = R\sum_{n=1}^{\infty}-e^{2in\theta}\left\{\frac{2}{2n+1}+\frac{2}{2n-1}-\frac{1}{n}-\frac{1}{n+1}+(-1)^n\left[\frac{1}{n}-\frac{1}{n+1}\right]\right\}$$
$$= R\left\{(1+e^{-2i\theta})\log\frac{1+e^{2i\theta}}{1-e^{2i\theta}}-2\cos\theta\log\frac{1+e^{i\theta}}{1-e^{i\theta}}\right\}$$
$$= \frac{\pi}{2}\sin 2\theta+2\cos\theta\log\tan\frac{\theta}{2}-2\cos^2\theta\log\tan\theta.$$

Hence we can write $L'_{21} = L'_{211}+L'_{212}$, where

$$L'_{211} = (c^5/2)\int_0^{\pi/2}(2\cos\theta)^{\frac{1}{2}}\sin^2\frac{\theta}{2}\sin\frac{3\theta}{2}\sin 2\theta\,d\theta$$
$$= 2^{-\frac{5}{2}}c^5\{I_{\frac{1}{2},\frac{1}{2}}-I_{\frac{3}{2},\frac{1}{2}}+I_{\frac{3}{2},\frac{7}{2}}-I_{\frac{1}{2},\frac{7}{2}}\} = 3\pi c^5/128,$$

using the notation and results of (18·18), and

$$L'_{212} = (c^5/\pi)\int_0^{\pi/2}(2\cos\theta)^{\frac{1}{2}}\sin^2\frac{\theta}{2}\sin\frac{3\theta}{2}\left\{2\cos\theta\log\tan\frac{\theta}{2}-2\cos^2\theta\log\tan\theta\right\}d\theta.$$

With the substitution $\cos\theta = \tan^2\phi$, this becomes

$$L'_{212} = (c^5/\pi)\Big[J_1 - J_2\Big]_0^{\pi/4},$$

where
$$J_1 = \int 4s(t^6+t^8-2t^{10})\log\sin\phi\,d\phi,$$

$$J_2 = \int s(-t^4+3t^8-2t^{10})\log\cos 2\phi\,d\phi.$$

Proceeding as with I_1 and I_2, we find

$$\begin{aligned}J_2 = {} & \{-\tfrac{1}{5}st^9+\tfrac{3}{5}st^7-\tfrac{7}{10}st^5+\tfrac{5}{8}st^3-\tfrac{15}{16}st+\tfrac{49}{80}\sqrt{2}\}\log(1-\sqrt{2}\sin\phi)\\ & +\{-\tfrac{1}{5}st^9+\tfrac{3}{5}st^7-\tfrac{7}{10}st^5+\tfrac{5}{8}st^3-\tfrac{15}{16}st-\tfrac{49}{80}\sqrt{2}\}\log(1+\sqrt{2}\sin\phi)\\ & +\tfrac{1}{10}st^7-\tfrac{23}{60}st^5+\tfrac{187}{240}st^3-\tfrac{291}{160}st+\tfrac{683}{160}\log(s+t)-\tfrac{15}{16}\int\sec\phi\log\sec 2\phi\,d\phi\end{aligned}$$

or
$$\Big[J_2\Big]_0^{\pi/4} = \tfrac{683}{160}\log(1+\sqrt{2})-\tfrac{15}{16}K-\tfrac{49}{40}\sqrt{2}\log 2-\tfrac{127}{96}\sqrt{2}$$

and
$$\begin{aligned}J_1 = {} & \{-\tfrac{4}{5}st^9+\tfrac{7}{5}st^7-\tfrac{29}{30}st^5+\tfrac{29}{24}st^3-\tfrac{29}{16}st+\tfrac{29}{16}\log(s+t)\}\log\sin\phi\\ & +\tfrac{1}{10}st^7-\tfrac{7}{20}st^5+\tfrac{163}{240}st^3-\tfrac{779}{480}st+\tfrac{1649}{480}\log(s+t)-\tfrac{29}{16}\int t^{-1}\log(s+t)\,d\phi\end{aligned}$$

or
$$\Big[J_1\Big]_0^{\pi/4} = \tfrac{1649}{640}\log(1+\sqrt{2})+\tfrac{233}{480}\sqrt{2}\log 2-\tfrac{29}{16}K'-\tfrac{29}{32}\log 2\log(1+\sqrt{2})-\tfrac{191}{160}\sqrt{2}.$$

Collecting up the results we have

$$L'_{21} = (c^5/\pi)\left\{\tfrac{15}{16}K-\tfrac{29}{16}K'-\tfrac{5}{6}\log(1+\sqrt{2})-\tfrac{29}{32}\log 2\log(1+\sqrt{2}) + \tfrac{821}{480}\sqrt{2}\log 2+\tfrac{31}{240}\sqrt{2}+\frac{\pi^2}{64}\right\}.$$

Hence we find L'_2 and then using (7·4) and (19·11) we have

$$L_2 = (c^5/\pi)\left\{\tfrac{15}{16}K-\tfrac{29}{16}K'-\frac{7+8\sqrt{2}}{12}\log(1+\sqrt{2})-\tfrac{29}{32}\log 2\log(1+\sqrt{2}) +\tfrac{1}{3}+\tfrac{11}{240}\sqrt{2}+\tfrac{821}{480}\sqrt{2}\log 2+\tfrac{1}{2}[\log(1+\sqrt{2})]^2+\frac{\pi^2}{64}\right\}. \quad (19·27)$$

Numerical values of K and K'

With the substitution $\sin\phi = x$, we have

$$K' = \int_0^{\pi/4} \cot\phi \log(\sec\phi + \tan\phi)\, d\phi = \int_0^{1/\sqrt{2}} \frac{1}{2x} \log\frac{1+x}{1-x}\, dx$$

$$= \int_0^{1/\sqrt{2}} \left\{\sum_{n=0}^{\infty} \frac{x^{2n}}{2n+1}\right\} dx = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{1}{2^n(2n+1)^2},$$

which is readily computed, and we find

$$K' = \int_0^{\pi/4} \cot\phi \log(\sec\phi + \tan\phi)\, d\phi = 0{\cdot}75609968. \tag{19·28}$$

The same substitution gives

$$K = \int_0^{\pi/4} \sec\phi \log\sec 2\phi\, d\phi = -\int_0^{1/\sqrt{2}} \frac{\log(1-2x^2)}{1-x^2}\, dx.$$

If we put
$$-\frac{\log(1-2x^2)}{1-x^2} = \sum_{n=1}^{\infty} a_n x^{2n},$$

then
$$\sum_{n=1}^{\infty} a_n (x^{2n} - x^{2n+2}) = -\log(1-2x^2) = \sum_{n=1}^{\infty} 2^n \frac{x^{2n}}{n},$$

which gives
$$a_n = \frac{2}{1} + \frac{2^2}{2} + \frac{2^3}{3} + \ldots + \frac{2^n}{n},$$

whence
$$K = \sum_{n=1}^{\infty} a_n \int_0^{1/\sqrt{2}} x^{2n}\, dx = \sum_{n=1}^{\infty} \frac{a_n}{(2n+1)\, 2^{n+\frac{1}{2}}}.$$

This, however, is not sufficiently rapidly convergent for purposes of computation, but since it is a series of positive terms we can certainly rearrange it as

$$K = \frac{1}{\sqrt{2}} \left\{\sum_{n=1}^{\infty} \frac{1}{n(2n+1)} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n(2n+3)} + \frac{1}{2^2} \sum_{n=1}^{\infty} \frac{1}{n(2n+5)} + \ldots\right\}.$$

Now
$$\sum_{n=1}^{\infty} \frac{1}{n(2n+2m-1)} = \frac{1}{2m-1} \sum_{n=0}^{\infty} \left\{\frac{1}{n+1} - \frac{1}{n+m+\frac{1}{2}}\right\}$$
$$= \frac{1}{2m-1} \{\psi(m+\tfrac{1}{2}) + \gamma\},$$

using the digamma function again, γ being Euler's constant (see (15·25)), whence

$$K = 2^{\frac{1}{2}} \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{1}{2m-1} \{\psi(m+\tfrac{1}{2}) + \gamma\}$$

from which K has been computed, and we find

$$K = \int_0^{\pi/4} \sec\phi \log\sec 2\phi\, d\phi = 0{\cdot}68856036. \tag{19·29}$$

With these values of K and K', equations (19·26) and (19·27) lead to the numerical values of M_1 and L_2 as

$$M_1 = 0{\cdot}0201000c^5, \quad L_2 = -0{\cdot}0019833c^5. \tag{19·30}$$

The associated twists and the centre of flexure

From the results of (19·11), with the notation of § 9, we find from (9·12) and (9·16) the numerical results

$$\left.\begin{aligned} A &= 0{\cdot}00867357c^4, \quad B = 0{\cdot}05459106c^4, \\ H &= 0{\cdot}00283615c^4, \quad C = 0{\cdot}04626652c^4, \end{aligned}\right\} \tag{19·31}$$

whence from (9·13) and (9·14)

$$\left.\begin{aligned} I &= 0{\cdot}05476557c^4, \quad \alpha = 3^\circ{\cdot}5205638, \\ I' &= 0{\cdot}00849906c^4, \end{aligned}\right\} \tag{19·32}$$

and from (9·17) we find

$$a_0 = 9{\cdot}3172895, \quad b_0 = 58{\cdot}642602, \quad h_0 = 3{\cdot}0466362. \tag{19·33}$$

Using these numerical results and those for the six moment integrals from (19·12), (19·17), (19·20), (19·24) and (19·30), also using the length a of the straight boundary instead of c, where $a = c\sqrt{2}$, we have from (7·7) and (7·8)

$$\tau = \quad (Wa/EI)\{0{\cdot}1707208 + 0{\cdot}1812218\eta\}, \tag{19·34}$$

$$\tau' = -(W'a/EI')\{0{\cdot}0204721 + 0{\cdot}0413880\eta\}. \tag{19·35}$$

Also from (9·5) and (9·6) we find the co-ordinates (f_0, g_0) of the centre of flexure as

$$f_0/a = 0{\cdot}5826902 + 0{\cdot}0301770\sigma, \tag{19·36}$$

$$g_0/a = 0{\cdot}1851923 + 0\ 0042282\sigma. \tag{19·37}$$

The values of the co-ordinates of the centroid in terms of a are

$$h/a = 0{\cdot}5553604, \qquad k/a = 0{\cdot}1449459 \tag{19·38}$$

for comparison.

Whatever the load-point and direction of the load, the general twist of the cross-section can be visualized from the relative position of the load-point and the centre of flexure.

20. Comparison of results

In this concluding section we group together certain of the results from previous sections for a comparison of the twisting effect of the load in the various cases. The sense of the mean twist is known at once from the sign of the moment of the load about the centre of flexure. Young, Elderton and Pearson envisaged the problem in the classical manner with the load at the centroid of the section and considered the twist as given by the sign of their "total torsion" so-called, as defined in § 13. They were rather disturbed to find results of opposite sign for the quadrantal circular sector and the lemniscate loop, and were not sure this could be correct as they knew of no other sections exhibiting what they termed "negative torsion". Consider fig. 8, which represents (*a*) the semi-circular cross-section, (*b*) the right-angled isosceles triangular cross-section, (*c*) the quadrantal circular sector cross-section, and (*d*) the lemniscate loop cross-section, so that we have a progression from (*a*) where the axis of symmetry is the shortest dimension to (*d*) where it is the longest dimension, the cross-sections all having the same axes of symmetry. For these cross-sections the quantity $(h-f_0)/a$, which is proportional to the associated twist τ', takes the values

$$
\begin{array}{lll}
(a) & \{0\cdot084845+0\cdot008972\sigma\} & >0 \text{ for all values of } \sigma, \\
(b) & \{0\cdot066666+0\cdot008717\sigma\} & >0 \text{ for all values of } \sigma, \\
(c) & \{0\cdot002572-0\cdot000232\sigma\} & >0 \text{ for all values of } \sigma, \\
(d) & -\{0\cdot015597+0\cdot011204\sigma\} & <0 \text{ for all values of } \sigma,
\end{array}
$$

so that for the associated twist τ' we do have a change of sign as between the cross-sections (*c*) and (*d*). Using Young, Elderton and Pearson's "total torsion", these are proportional to the quantities

$$
\begin{array}{lll}
(a) & \{0\cdot224047+0\cdot172141\eta\} & >0 \text{ for all possible values of } \eta, \\
(b) & \{0\cdot212941-0\cdot138823\eta\} & >0 \text{ for all possible values of } \eta, \\
(c) & \{0\cdot004458-0\cdot096155\eta\} & <0 \text{ for } \eta>0\cdot0464, \\
(d) & -\{0\cdot012447+0\cdot076748\eta\} & <0 \text{ for all values of } \eta.
\end{array}
$$

Here the change of sign occurs, for practical values of η, as between the cross-sections (*b*) and (*c*), and not between (*c*) and (*d*) as Young, Elderton and Pearson incorrectly found. There is no anomaly in a change of sign for this quantity for cross-sections not markedly different in character, as they were inclined to suggest; it simply indicates a passage of the centre of flexure through the centroid, as the uni-axial cross-sections change their shape in the manner of the series of cross-sections illustrated.

Compare also the positions of the centres of flexure of the cross-sections grouped together in fig. 9. Of these, the uni-axial cross-sections (*a*) and (*c*) are right-angled isosceles triangles, (*b*) is a 45° circular sector, also a uni-axial section, and (*d*) is the asymmetric cross-section considered in § 19, the half loop of the lemniscate of Bernoulli.

The co-ordinates of the flexural centres in these cases are given by

$$
\begin{array}{ll}
(a)\ f_0/a = 0\cdot7\quad +0\cdot004\sigma, & g_0/a = 0\cdot3\quad -0\cdot004\sigma, \\
(b)\ f_0/a = 0\cdot645+0\cdot026\sigma, & g_0/a = 0\cdot267+0\cdot011\sigma, \\
(c)\ f_0/a = 0\cdot5, & g_0/a = 0\cdot2\quad +0\cdot004\sigma, \\
(d)\ f_0/a = 0\cdot583+0\cdot030\sigma, & g_0/a = 0\cdot185+0\cdot004\sigma,
\end{array}
$$

whereas the co-ordinates of the centroids are given by

$$\begin{array}{lll} (a) & h/a = 0{\cdot}667, & k/a = 0{\cdot}333, \\ (b) & h/a = 0{\cdot}600, & k/a = 0{\cdot}249, \\ (c) & h/a = 0{\cdot}5, & k/a = 0{\cdot}167, \\ (d) & h/a = 0{\cdot}555, & k/a = 0{\cdot}145. \end{array}$$

In these compact sections the centroid and centre of flexure are not far apart, so that, for such sections, the usual engineering practice of taking them as coincident involves a relatively small error. But for less compact sections, as, for example, the circular section with sectorial notch (see § 15, Table I, columns 6 and 7), their distance apart increases, being a maximum in this particular case when the notch reduces to a radial slit. For non-closed thin-walled sections it is well known from elementary approximate theory (Timoshenko 1931, p. 195) that these points may be well outside the section and on opposite sides of it.

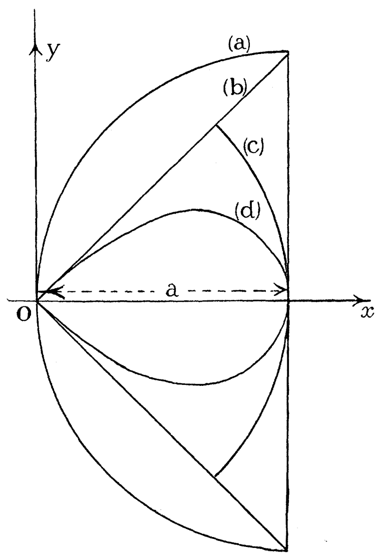

FIG. 8

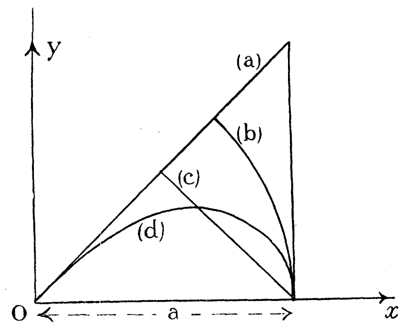

FIG. 9

FIG. 8. Cross-sections contained within the semi-circle, to illustrate discussion of the passage of the centre of flexure through the centroid of the cross-section as the type of cross-section changes.

FIG. 9. Cross-sections contained within right-angled isosceles triangle, to illustrate discussion of change of position of the centre of flexure as the type of cross-section changes.

21. CONCLUDING REMARKS

Young, Elderton and Pearson's work on the circular section with complete radial slit, and the split tube, and of Shepherd on the circular section with radial slit of any depth, and the cardioid section give us an idea of the effects of longitudinal cracks or slits on the twisting of a beam under flexure. Until quite recently we had no corresponding information of the effect of holes and notches. The writer has considered several problems of this type with the aid of the canonical flexure functions and hopes to publish further results in due course. One of these problems, that of the circular beam with an eccentric circular cylindrical hollow, suggested by Love (1906, p. 325), as a soluble problem, and again by Young, Elderton and Pearson (1918, p. 69), has been solved by Saint-Venant's classical flexure functions and published by Seth (1936 *b*, 1937) who gives some numerical results for the amount of twist produced in the case where the load is at right angles to the line of centres and for the value of stress at different points of the section. This is a valuable cross-section and some information as to the position of the centre of flexure for a range of sizes of cavity and position of the cavity seems desirable. The writer hopes to compare his solution with Seth's later. The torsion function for the problem was first discovered by Macdonald (1893).

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After the writer had discovered the simple canonical boundary conditions covering the flexure problem for any origin and axes, and had corrected the erroneous solution of the problem of the lemniscate loop, it was suggested to him by Professor L. N. G. Filon that the work would be rendered more compact by making use of the complex variable. Not only did it do this very satisfactorily but it led the writer to the method of § 14 giving the results for the lemniscate loop with a tremendous saving of labour compared with that of his first solution, and the writer's thanks are due to Professor Filon for orienting his thoughts in this direction and for his ever helpful interest in discussing the problem.

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